PROBLEM SET 5: RANDOM PROBLEMS
DUE: FEBRUARY 11

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Problem 1. *** Let there be given nine lattice points in three-dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

Problem 2. *** Let $f(x)$ be a polynomial of degree n with real coefficients and such that $f(x) \geq 0$ for every real number x. Show that

$$f(x) + f'(x) + f''(x) + \cdots + f^{(n)}(x) \geq 0$$

for all real x. ($f^{(k)}(x)$ is the k-th derivative.)

Problem 3. ** The Euclidean plane is divided into regions by drawing a finite number of straight lines. Show that it is possible to color each of these regions either red or blue in such a way that no two adjacent regions have the same color.

Problem 4. ** Show that there are no positive integers x, y such that $y^2 = x^2 + x + 1$.

Problem 5. ** Let a, b be nonzero numbers with

$$a + b = \frac{1}{a} + \frac{1}{b}.$$

Prove that

$$a^3 + b^3 = \frac{1}{a^3} + \frac{1}{b^3}.$$

Problem 6. *** Prove that, if a pentagon inscribed in a circle has equal angles, then its sides are equal.

Problem 7. * Let $a_1, a_2, \ldots, a_n$ represent an arbitrary arrangement of the numbers $1, 2, \ldots, n$. Prove that, if n is odd, the product

$$(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$$

is an even number.

Problem 8. ** Let α be real, and not an odd multiple of π. Prove that $\tan(\alpha/2)$ is rational if and only if both $\cos(\alpha)$ and $\sin(\alpha)$ are rational.

Problem 9. ***** Suppose that n is a positive integer. Write down all the rational numbers $\frac{a}{b}$ with $\gcd(a, b) = 1$ and $0 \leq a \leq b \leq n$, arranged from small to large.

Prove that

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$$

are three consecutive numbers in this sequence, then

$$\frac{a_2}{b_2} = \frac{a_1 + a_3}{b_1 + b_3}.$$

For example, for $n = 5$ we have

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

and

$$\frac{1}{3} = \frac{1+2}{4+5}.$$

Problem 10. ******** Suppose that f is a continuous real-valued function on the real numbers. Suppose that f^2 and f^3 are both $\mathcal{C}^\infty$ (which means that the k-th derivative exists and is continuous for all $k = 0, 1, 2, \ldots$). Prove that f is $\mathcal{C}^\infty$.