PROBLEM SET 4: PIGEONHOLE PRINCIPLE
DUE: FEBRUARY 4

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The pigeonhole principle is the following observation:

Theorem 1. *Suppose that $> kn$ marbles are distributed over n jars, then one jar will contain at least $\geq k+1$ marbles.*

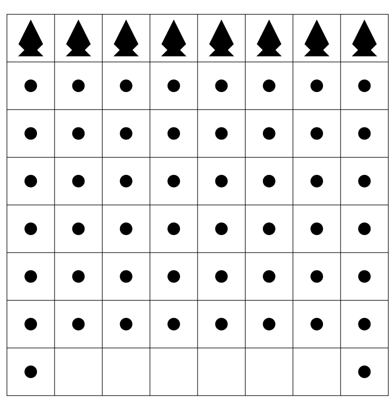
(It can also be formulated in terms of pigeons and pigeonholes, hence the name.) The proof of this pigeonhole principle is easy. It is more difficult to know when to apply it. There are many surprising applications of the pigeonhole principle. The pigeonhole principle was first explicitly formulated by the mathematician Dirichlet (1805–1859).

The pigeonhole principle says for example that at least two people in New York City will have the same number of hairs on their head. This is because humans have $< 1,000,000$ hairs and there are $> 1,000,000$ people in NYC.

The pigeonhole principle is particularly powerful in existence proofs which are not constructive. For example in the previous example we proved the existence of two people with the same number of hairs without specifically identifying these two individuals.

Example 1. How many bishops can one put on an 8×8 chessboard such that no two bishops can hit each other.

Discussion. It seems like a good idea to put 8 bishops in one row at the edge of the board. Let us see which fields these bishops can reach:



There are only 6 fields that cannot be reached by the 8 bishops. If we put 6 bishops on these positions then we see that still no two bishops can hit each other. So we see that at least 14 bishops can be put on the chessboard. We conjecture that this number is maximal.

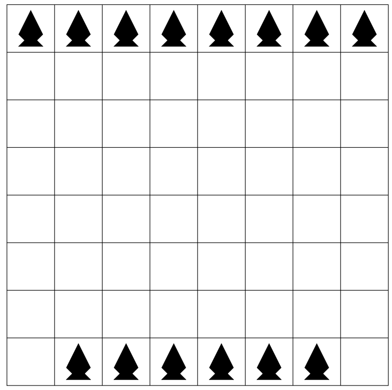
How can we prove that 14 is the maximal number of bishops? We can use the pigeonhole principle. We need to partition the $8 \times 8 = 64$ fields into 14 sets such that whenever two bishops are on fields which lie in the same set, then they can hit each other.

Note that a bishop on a black field only can move to other black fields. A bishop on a white field can only move to other white fields. To prove that 14 is the maximal number of bishops, we could prove that there at most 7 "black" bishops and at most 7 "white" bishops. We try to partition the 32 black fields into 7 sets such that if two bishops are on fields in the same set, then they can hit each other. We see that the following configuration works:

	1		2		3		4
1		2		3		4	
	2		3		4		5
2		3		4		5	
	3		4		5		6
3		4		5		6	
	4		5		6		7
4		5		6		7	

A similar configuration works for the white fields. (Take the mirror image.) In the proof that we are going to write down, we do not need to distinguish between "black" and "white" bishops. We can combine the partition of black fields and the partition of the white fields to get a partition of the set of all 64 fields into 14 subsets.

Proof. The answer is 14. Place 14 bishops on the chessboard as follows:



Then no two bishops can hit each other.

Let us label the fields on the chessboard as follows:

11	1	10	2	9	3	8	4
1	11	2	10	3	9	4	8
12	2	11	3	10	4	9	5
2	12	3	11	4	10	5	9
13	3	12	4	11	5	10	6
3	13	4	12	5	11	6	10
14	4	13	5	12	6	11	7
4	14	5	13	6	12	7	11

Whenever two bishops are placed on the chessboard with the same number, they can hit each other. If a number of bishops are placed on fields of the chessboard such that no two can hit each other, then all the numbers of the fields are distinct. This shows that the number of bishops is at most 14. ☺

Example 2. (Putnam 1958) Let S be a subset of $\{1, 2, 3, \ldots, 2n\}$ with $n + 1$ elements. Show that one can choose distinct elements $a, b \in S$ such that a divides b.

Discussion. Is the $n + 1$ in the problem sharp? Suppose that S has the property that for every pair of distinct $a, b \in S$, a does not divide b and b does not divide a. How many elements can S have?

If S contains 1 then it could not contain any other element. If S contains 2 then all other even numbers are excluded. In order to get S as large as possible, it seems that one should choose large elements. If we take the n largest elements, $S = \{n+1, n+2, \ldots, 2n\}$ then clearly no two distinct elements divide each other.

Any positive integer a with $a \leq n$ divides an element of S. This shows that S is maximal with the desired property.

The problem has somewhat of a pigeonhole flavor: We are asked to prove the existence of certain elements $a, b \in S$ but it seems unlikely that we can explicitly construct these elements.

How can we apply the pigeon hole principle? Since we have $n+1$ elements we partition $\{1, 2, \ldots, 2n\}$ into n sets $T_1, T_2, \ldots, T_n$. The pigeonhole principle says that $S \cap T_i$ contains at least two elements for some i. This then should be useful to conclude that there exist $a, b \in S$ such that a divides b. So we would like T_i to have the following property: Whenever $a, b \in T_i$ with $a < b$ then a divides b. So T_i should be of the form

$$\{a_1, a_2, a_3, \ldots, a_k\}$$

with $a_1 \mid a_2 \mid a_3 \mid \cdots \mid a_k$. The largest such set is

$$\{1, 2, 2^2, 2^3, \ldots\}.$$

Let us define T_1 to be this set. The smallest element not in T_1 is 3. So let us define

$$T_2 = \{3, 3 \cdot 2, 3 \cdot 2^2, 3 \cdot 2^3, \ldots\}.$$

The smallest element, not in T_1 and T_2 is 5. so let us define

$$T_3 = \{5, 5 \cdot 2, 5 \cdot 2^2, 5 \cdot 2^3, \ldots\}.$$

Let us define more generally

$$T_k = \{(2k-1), (2k-1) \cdot 2, (2k-1) \cdot 2^2, (2k-1) \cdot 2^3, \ldots\}$$

for $k = 1, 2, \ldots, n$. It is easy to see that the union of $T_1, T_2, \ldots, T_n$ contains. $\{1, 2, \ldots, 2n\}$. We are now ready to write down the proof, using the pigeonhole principle.

Proof. Let us define

$$T_k = \{1, 2, \ldots, 2n\} \cap \{(2k-1), (2k-1) \cdot 2, (2k-1) \cdot 2^2, (2k-1) \cdot 2^3, \ldots\}$$

for all $k = 1, 2, \ldots, n$. Every $c \in \{1, 2, \ldots, 2n\}$ can uniquely be written as $c = 2^j(2i-1)$ with $j \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}$. This shows $T_1, T_2, \ldots, T_n$ is a partition of $\{1, 2, \ldots, 2n\}$. By the pigeon hole principle, $S \cap T_i$ contains at least two distinct elements for some i, say $\{a, b\} \subset S \cap T_i$ with $a < b$. then $a = 2^j(2i-1)$ and $b = 2^k(2i-1)$ for some $j, k \in \mathbb{N}$ with $j < k$ and it is clear that $a \mid b$. ☺

Example 3. Suppose that S is a set of n integers. Show that one can choose a nonempty subset T of S such that the sum of all elements of T is divisible by n.

Discussion. One can try to attack this problem using the pigeonhole principle but it is not immediately clear how we can apply it. What are the "pigeonholes" here? Since our goal is to prove that something is divisible by n, it is natural to take the congruence classes modulo n as pigeonholes. The pigeonhole principle says: "Given $n+1$ integers, one can choose two of them such that their difference is divisible by n." How can we apply this here? We can apply it if we have integers $a_1, a_2, \ldots, a_{n+1}$ such that $a_j - a_i$ is a sum of distinct elements of S for all $i < j$. We can indeed achieve this. Suppose that $S = \{b_1, b_2, \ldots, b_n\}$. Take $a_1 = 0$, $a_2 = b_1$, $a_3 = b_1 + b_2$, $a_4 = b_1 + b_2 + b_3$, etc. We can write down our proof:

Proof. Suppose that $S = \{b_1, b_2, \ldots, b_n\}$. Define

$$c_i = b_1 + b_2 + \cdots + b_i$$

for all $i = 0, 1, \ldots, n$ ($c_0 = 0$). Of the $n+1$ numbers $c_0, c_1, \ldots, c_n$, at least 2 must lie in the same congruence class module n by the pigeonhole principle. Assume that we have $c_i \equiv c_j (\mathrm{mod} n)$ for $i < j$. Then we get that

$$c_j - c_i = b_{i+1} + b_{i+2} + \cdots + b_j$$

is divisible by n. ☺

Example 4. Suppose that we are given a sequence of $nm + 1$ distinct real numbers. Prove that there is an increasing subsequence of length $n+1$ or a decreasing subsequence of length $m + 1$.

Discussion. To get an idea, let us do a random example with $n = m = 3$:

(1) $$55, 63, 57, 60, 74, 85, 16, 61, 7, 49.$$

What is the longest increasing subsequence and what is the longest decreasing subsequence? How can we efficiently find these without checking all possible subsequences?

For a longest decreasing sequence in (1) there are two cases. Either such a sequence ends with 49 or it does not. If the sequence does not contain 49, then a longest decreasing sequence of (1) is a longest decreasing subsequence of the shortened sequence

(2) $$55, 63, 57, 60, 74, 85, 16, 61, 7.$$

If 49 appears in a longest decreasing subsequence, then we may wonder what the previous element in that subsequence is. Is it 55,63,57,60,74,85 or 61? (Clearly 16 and 7 are out of the question because the subsequence is decreasing.) It would now be useful to know for each x in $\{55, 63, 57, 60, 74, 85, 61\}$ what the longest decreasing subsequence is of (2) that ends with x. We then could take the longest decreasing subsequence of (2) ending with some x in $\{55, 63, 57, 60, 74, 85, 61\}$ and add 49 at the end to obtain a longest decreasing subsequence of (1).

So to find the longest decreasing subsequence of (1) we need to find the longest decreasing subsequence of (1) ending with x for $x = 55, 63, \ldots, 49$ (in that order).

x	a longest decreasing subsequence ending with x
55	55
63	63
57	$63, 57$
60	$63, 60$
74	74
85	85
16	$63, 60, 16$
61	$85, 61$
7	$63, 60, 16, 7$
49	$85, 61, 49$

We have found that the longest decreasing subsequence has length 4. Namely the subsequence 63,60,16,7 is decreasing. So the statement we want to prove works out in this example. Let us determine the longest increasing sequence:

x	a longest decreasing subsequence ending with x
55	55
63	$55, 63$
57	$55, 57$
60	$55, 57, 60$
74	$55, 57, 60, 74$
85	$55, 57, 60, 74, 85$
16	16
61	$55, 57, 60, 61$
7	7
49	$7, 49$

We see that there even is an increasing sequence of length 5.

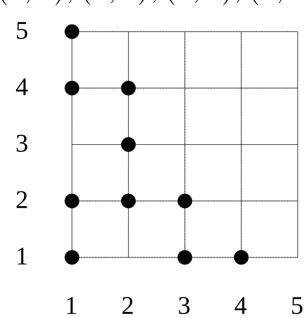
We checked one (small) example and it seems that we are still far from a solution (but this is actually not the case).

We are interested in the lengths of maximal increasing/decreasing sequences. So let us make a table containing the length of a longest increasing sequence ending in x and the length of a longest decreasing sequence ending in x for all x.

x	decreasing	increasing
55	1	1
63	1	2
57	2	2
60	2	3
74	1	4
85	1	5
16	3	1
61	2	4
7	4	1
49	3	2

Let us plot the last two columns against each other. We get 10 *distinct* points

$$(1,1),(1,2),(2,2),(2,3),(1,4),(1,5),(3,1),(2,4),(4,1),(3,2).$$



If there were no increasing or decreasing sequence of length 4, then all points would fit in a 3×3 box and two of the points would have to coincide by the pigeonhole principle. This leads to a contradiction as the following proof shows.

Proof. Suppose that

$$x_1, x_2, \ldots, x_{mn+1}$$

is a sequence of distinct real numbers. Let a_i be the length of the longest decreasing subsequence ending with x_i. Let b_i be the length of the longest increasing subsequence ending with x_i. We claim that if $i \neq j$, then that $(a_i, b_i) \neq (a_j, b_j)$.

Indeed, if $x_j < x_j$ then we can take a longest decreasing subsequence ending with x_i and add x_j at the end. This way we obtain a decreasing subsequence ending with x_j of length $a_i + 1$. This shows that $a_j > a_i$ and $(a_i, b_i) \neq (a_j, b_j)$.

If $x_j > x_i$ then we can take a longest increasing subsequence ending with x_i and add x_j at the end. This shows that $b_j > b_i$ and $(a_i, b_i) \neq (a_j, b_j)$.

There are only nm pairs (a, b) with $a, b \in \mathbb{Z}$ and $1 \leq a \leq m$ and $1 \leq b \leq n$. We must have $a_i > m$ or $b_i > n$ for some i. ☺

Problem 1. * What is the maximum number of rooks that one place on an 8×8 chessboard such that no two rooks can hit each other? Prove your answer.

Problem 2. * What is the maximum number of knights that one place on an 8×8 chessboard such that no two knights can hit each other? Prove your answer.

Problem 3. **** What is the maximum number of Queens that one can place on an 8×8 chessboard such that no two Queens can hit each other?

Problem 4. **** [Dutch Mathematical Olympiad] A set S of positive integers is called *square-free* if for all distinct $a, b \in S$ we have that the product ab is not a square. What is the maximum cardinality of a square free subset $S \subseteq \{1, 2, 3, \ldots, 25\}$?

Problem 5. * Show that $(a-b)(a-c)(b-c)$ is always even if a, b, c are integers.

Problem 6. **** Prove that for every integer $n \geq 2$ there exists an integer m such that $k^3 - k + m$ is not divisible by n for all integers k.

Problem 7. *** Show that, given a 7-digit number, you can cross out some digits at the beginning and at the end such that the remaining number is divisible by 7. For example, if we take the number 1234589, then we can cross out 1 at the beginning and 89 at the end to get the number 2345=7 $\times$ 335.

1. Diomphantine Approximation

Example 5. Suppose that α is a real number and that N is a positive integer. Show that one of the numbers $\alpha, 2\alpha, 3\alpha, \ldots, N\alpha$ differs at most $\frac{1}{N}$ from an integer.

Discussion. It is not so clear a priori how one can use the pigeonhole priciple in this problem. Since we are only interested in the value $i\alpha$ modulo the integers, we define $\beta_i = i\alpha - \lfloor i\alpha \rfloor$. ($\lfloor x \rfloor$ is the largest integer $\leq x$.) We observe that $0 \leq \beta_i < 1$ for all i. We want to show that $\beta_i \leq \frac{1}{N}$ or $\beta_i \geq 1 - \frac{1}{N}$ for some i.

An important observation is that $\beta_{i+j} - \beta_i$ is the same as α_j up to an integer for all i. So if β_{i+j} is very close to β_i then α_j is very close to an integer. We want to show that one can choose two of the $N+1$ numbers $\beta_0 = 0, \beta_1, \beta_2, \ldots, \beta_n$ that are at most of distance $\frac{1}{N}$ of each other.

Here is where the pigeonhole principle might come in. We need to partition the interval $[0, 1\rangle$ into N sets, such that every two elements in the same set are at most $\frac{1}{N}$ apart. This is possible. We can take $S_k = [\frac{k-1}{N}, \frac{k}{N}\rangle$ for $k = 1, 2, \ldots, N$. By the pigeon hole principle at least two of the numbers in $\beta_0, \beta_1, \ldots, \beta_N$ lie in the same set S_k. Let us write down a more formal proof.

Proof. Define

$$\beta_i = i\alpha - \lfloor i\alpha \rfloor$$

for $i = 0, 1, 2, \ldots, N$. Then $0 \leq \beta_i < 1$ for all i. Define $S_k = [\frac{k-1}{N}, \frac{k}{N}\rangle$ for $k = 1, 2, \ldots, N$. The interval $[0, 1\rangle$ is the union of intervals $S_1, S_2, \ldots, S_N$. By the pigeon hole principle, at least two of the numbers $\beta_0, \beta_1, \ldots, \beta_N$ lie in the same interval S_k for some k. Say $\beta_i, \beta_j \in S_k$ for some i, j with $0 \leq i < j \leq N$. But then

$$-\frac{1}{N} \leq \beta_j - \beta_i = (j-i)\alpha - (\lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor) \leq \frac{1}{N}.$$

We are done because $1 \leq j - i \leq N$ and $\lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor$ is an integer. ☺

Diomphantine approximation is an area of number theory where one likes to approximate irrational numbers by rational number. The previous example allows us to characterize irrational numbers!

Theorem 2. *An real number α is irrational if and only if there exists a sequences of integers $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ such that*

$$\lim_{n\to\infty} q_n\alpha - p_n = 0$$

and $q_n\alpha - p_n \neq 0$ for all n.

Proof. Suppose that α is irrational. For every n, we can find integers p_n, q_n with $1 \leq q_n \leq n$ such that $|q_n\alpha - p_n| < \frac{1}{n}$. It follows that

$$\lim_{n\to\infty} q_n\alpha - p_n = 0.$$

Obviously $q_n\alpha - p_n \neq 0$ for all n because α is irrational.

Conversely, suppose that $q_n\alpha - p_n \neq 0$ for all n and $\lim_{n\to\infty} q_n\alpha - p_n = 0$. Assume that α is rational, say $\alpha = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b > 0$. Now we have

$$|q_n\alpha - p_n| = \left|\frac{q_n a - p_n b}{b}\right| \geq \frac{1}{b}$$

because $q_n\alpha - p_n \neq 0$. This leads to a contradiction with $\lim_{n\to\infty} q_n\alpha - p_n = 0$. ☺

We can apply this:

Theorem 3. *The number $\sqrt{2}$ is irrational.*

Proof. Expanding $(\sqrt{2} - 1)^n$ and using $(\sqrt{2})^2 = 2$ we get

$$(\sqrt{2} - 1)^n = q_n\sqrt{2} - p_n$$

for some integers p_n and q_n. We have

$$\lim_{n\to\infty} q_n\sqrt{2} - p_n = \lim_{n\to\infty} (\sqrt{2} - 1)^n = 0$$

because $|\sqrt{2} - 1| < 1$. Also

$$q_n\sqrt{2} - p_n = (\sqrt{2} - 1)^n \neq 0$$

for all n because $\sqrt{2} \neq 1$. This proves that $\sqrt{2}$ is irrational. ☺

Problem 8. *** Prove that the Euler number

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

is irrational.

Problem 9. **** Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$ with $|x_i| \leq 1$ for $i = 1, 2, \ldots, n$. Show that there exist $a_1, a_2, \ldots, a_n \in \{-1, 0, 1\}$, not all equal to 0, such that

$$|a_1x_1 + a_2x_2 + \cdots + a_nx_n| \leq \frac{n}{2^n - 1}$$

Problem 10. ***** [IMO 1987] Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all 0, such that $|a_i| \leq k - 1$ for all i and

$$|a_1x_1 + a_2x_2 + \cdots + a_nx_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

2. BALLS

For a finite set S we denote the number of elements of S by $|S|$. The pigeonhole principle can be reformulated as:

Theorem 4. *If $S_1, S_2, \ldots, S_n$ are subsets of a finite set T and*

$$|S_1| + |S_2| + \cdots + |S_n| > k|T|$$

then there exists an element $x \in T$ that lies in at least $k+1$ of the sets $S_1, S_2, \ldots, S_n$.

If S is a measurable set in $\mathbb{R}^3$, let $\mu(S)$ be its volume. We have the following variation of the previous theorem.

Theorem 5. *If $S_1, S_2, \ldots, S_n$ are measurable subsets of a measurable set T, and*

$$\mu(S_1) + \mu(S_2) + \cdots \mu(S_n) > k\mu(T).$$

then there exists an element $x \in T$ that lies in at least $k+1$ of the sets $S_1, S_2, \ldots, S_n$.

Example 6. Let S be a set of points in the cube $[0, 1] \times [0, 1] \times [0, 1]$ (in $\mathbb{R}^3$). Such that the distance between every two distinct elements $x, y \in S$ is at least 0.1. Give an upper bound for the number of elements of S.

Discussion. We could use the pigeonhole principle as follows. Partition the cube into small regions, such that in each region the maximal distance between two points in this region is < 0.1. For example, we could partition the cube into $N \times N \times N$ little cubes of sidelength $\frac{1}{N}$. The maximum distance between two points in this little cube is $\sqrt{3}/N$, the length of its diagonal. We need that $\sqrt{3}/N < 0.1$, so $N > 10\sqrt{3} \approx 17.32$. (Without a calculator we see that $10\sqrt{3} < 18$ because $18^2 = 324 > 300 = (10\sqrt{3})^2$.) So let us take $N = 18$. Each of the $N \times N \times N$ cubes can contain at most 1 element of S. Therefore, the cardinality of S is at most $18^3 = 324 \cdot 18 = 5832$.

There is another way of looking at this problem. Instead of saying that two points x and y have distance at least 0.1, we could say that the balls with radius 0.05 around x and around y are disjoint. Note that all balls lie within the cube $[-0.05, 1.05] \times [-0.05, 1.05] \times [-0.05, 1.05]$ with volume $1.1^3 = 1.331$. We can reformulate the problem as the problem of packing oranges of diameter 0.1 into a (cube-shaped) box of sidelength 1.1. The volume of a ball with radius 0.05 is $\frac{4}{3}\pi(0.05)^3 \approx 0.00523599$ (we are using a calculator now). The number of oranges is at most

$$\frac{(1.1)^3}{\frac{4}{3}\pi(0.05)^3} = \frac{3\cdot 22^3}{4\pi} = \frac{7986}{\pi} \approx 2542.02.$$

This means that S has at most 2542 points. This is quite an improvement.

Johannes Kepler (1571–1630) conjectured in 1611 that the densest way of packing balls in $\mathbb{R}^3$ is the cubic or hexagonal packing (well-known to people selling oranges). These packing give a density of

$$\frac{\pi}{3\sqrt{2}} \approx 74.048\%.$$

Kepler's conjecture was proven by Thomas Hales (U of M!) in 1998.

Using this result, we see that S can have at most

$$\frac{\pi}{3\sqrt{2}}\frac{7986}{\pi} = 1331\sqrt{2} \approx 1882.3183.$$

So S can have at most 1882 elements. (We do not claim here that this number is sharp.)

Example 7. A binary word of length n is a sequence of 0's and 1's of length n. The set $\{0,1\}^n$ is the set of all binary words of length n. Let S be a subset of $\{0,1\}^n$ with the following property: for every pair of distinct elements $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ we have that x and y differ in at least 3 positions. Show that S has at most

$$\frac{2^n}{n+1}$$

elements.

Proof. We can use the ideas in the previous example. The *distance* $d(x,y)$ between two words $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ is the number of positions where they differ. For any binary word $x \in \{0,1\}^n$, let $B(x)$ be the *ball with radius 1*, so

$$B(x) = \{z \in \{0,1\}^n \mid d(x,z) \leq 1\}.$$

In other words $B(x)$ is the set of all binary words of length n which differ from x in at most 1 position. The number of elements of $B(x)$ is $n+1$ (namely x itself and all words obtained by changing x at on position). Notice now that $d(x,y) \geq 3$ is equivalent with $B(x)$ and $B(y)$ are disjoint. So all balls

$$B(x), \quad x \in S.$$

are disjoint. The disjoint union of all balls

$$\bigcup_{x\in S} B(x)$$

has exactly $|S|(n+1)$ elements. On the other hand, this union is a subset of $\{0,1\}^n$ which has 2^n elements. We obtain the inequality

$$|S|(n+1) \leq 2^n$$

☺

The previous example is known as the *Hamming bound for* 1*-error correcing binary codes*. The mathematician Richard Hamming (1915–1998) also studied examples where equality holds (and these are known as Hamming codes).

Problem 11. **** Improve the statement in Example 5: Suppose that α is a real number and that N is a positive integer. Show that one of the numbers $\alpha, 2\alpha, 3\alpha, \cdots, N\alpha$ differs at most $\frac{1}{N+1}$ from an integer.

3. Extra Problems

Problem 12. ***** [Dutch Mathematical Olympiad] Suppose that S is a subset of $\{1,2,3,\ldots,30\}$ with at least 11 elements. Show that one can choose a nonempty subset T of S such that the product of all elements of T is a square.

Problem 13. *** Suppose that $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$ are integers. Show that the product

$$\prod_{0 \leq i < j \leq n} (a_j - a_i)$$

is divisible by $n!$.

Problem 14. **** Let $a_1, a_2, \ldots, a_{10}$ be distinct integers from $\{1,2,\ldots,99\}$. Show that $\{a_1, a_2, \ldots, a_{10}\}$ contains two disjoint non-empty subsets with the sum of the numbers from the first equal to the sum of the elements from the second subset.

Problem 15. **** [Putnam 2000] Let a_j, b_j, c_j be integers $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

Problem 16. ***** The set M consists of 2001 distinct positive integers, none of which is divisible by any prime $p > 23$. Prove that there are distinct x, y, z, t in M such that $xyzt = u^4$ for some integer u.

Problem 17. **** [Putnam 1985] Let m be a positive integer and let $\mathcal{G}$ be a regular $(2m+1)$-gon inscribed in the unit circle. Show that there is a postive constant A, independent of m, with the following property. For any point p inside $\mathcal{G}$ there are two distinct vertices v_1 and v_2 of $\mathcal{G}$ such that

$$||p - v_1| - |p - v_2|| < \frac{1}{m} - \frac{A}{m^2}.$$

Here $|s - t|$ denotes the distance between the points s and t.

Problem 18. *** Show that

$$\sum_p \frac{1}{2^p}$$

where p runs over all prime numbers, is an irrational number.