PROBLEM SET 3: ∞

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Choose 3 problems and hand them in next week, Wednesday, January 28. This problem set is about infinity. It may be more abstract than some other problem sets. I hope it is still enjoyable though. The idea of an infinite hotel was invented by Hilbert. Here is my own version of the story.

Once upon a time, there was a hotel that had an infinite number of rooms, the *Holiday Infinity*. The rooms were numbered 1,2,3,.... One day all the rooms were occupied in this hotel. Yet a new person appeared and asked for a room. The receptionist had a clever solution, namely every person in the hotel was asked to move "up" one room. In other words, the people occupying room n should move to room $n+1$. This of course freed up room number 1 which was given to the new guest. The day after this, the hotel next door, the *Comfort Infinity* burnt down. No one was hurt, but the comfort Infinity had infinitely many rooms, just as the Holiday Infinity, which were all occupied and all those infinitely many people needed a room. Again the clever receptionist found a solution: All the people in the Holiday infinity were asked to move to the room with the double of their current room number. This freed up all the rooms with an odd room number. Then the people who had room n in the *Comfort Infinity* were offered room number $2n-1$ in the Holiday Infinity and everyone had a place to sleep. People started to find out that one infinite hotel can house the people of two infinite hotels, and this led to an enourmous price battle between the different hotel chains which led to a sudden bankruptcy of the hotel chain the *Days Infinity*. The Days infinity chain consisted of infinitely many hotels, numbered 1,2,3,4,..., each with an inifinite number of rooms. The Days Infinity chain had no choice to throw out all of their guests in all of the infinitely many hotels at once. They all wanted to sleep at the Holiday Infinity but the receptionist of the Holiday Infinity resigned. (Nevertheless, there would have been a solution to give a room of all the guests from the Days Infinity chain.)

The previous story may give you the impression that all infinite sets have the same "size". We will give a more clear definition what it means for two sets to have the same size, and we will also see that there *are* different magnitudes of infinity. However, all the "infinities" in the previous story were all of the same infinite magnitude, called "countable". We will explain these notions. One of the well-known pioneers about infinity was the mathematician Cantor who lived in the nineteenth and early twentieth century.

We need a little bit of sets and functions.

If X, Y are sets we define the product set $X \times Y$ as the set of all pairs (x, y) with $x \in X$ and $y \in Y$.

A function $f : X \to Y$ is a "rule" that attaches to each element x of the set X and element $f(x)$ of the set Y.

Definition 1. A function $f : X \to Y$ is called *injective* or *one-to-one* if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. A function $f : X \to Y$ is called *surjective* or *onto* if and only if for every $y \in Y$ there exists an $x \in X$ such that $f(x) = y$. A function $f : X \to Y$ is called *bijective* if it is injective and surjective.

For example the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not injective and not surjective. It is not injective because $f(1) = f(-1)$ but $1 \neq -1$. It is not surjective because $f(x) = -1$ does not have a solution for $x \in \mathbb{R}$. The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3$ is bijective. Indeed, $g(x)$ is injective because if $x^3 = g(x) = g(y) = y^3$ then we must have $x = y$. The function $g(x)$ is surjective because every real number has a third root.

Definition 2. If $f : X \to Y$ is a function and $g : Y \to Z$ is a function, then we can define the composition function $g \circ f : X \to Z$ as follows. For every $x \in X$ we define

$$(g \circ f)(x) = g(f(x)).$$

Definition 3. Suppose that the function $f : X \to Y$ is bijective. Then one can define a function $f^{-1} : Y \to X$ as follows. For every $y \in Y$ there exists an $x \in X$ such that $f(x) = y$ because f is sujective. Moreover, x is unique because f is injective. Let us define $f^{-1}(y) = x$.

One can easily verify that f^{-1} is again bijective. We have $f^{-1}(f(x)) = x$ for all $x \in X$ (this follows from the definition). If $y \in Y$, then $f(f^{-1}(y)) = y$ (this also follows from the definition). For example, if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^3$ then f is bijective and $f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by $f^{-1}(x) = \sqrt[3]{x}$. Let $\mathbb{R}_+$ be the set of positive real numbers. The function $g : \mathbb{R} \to \mathbb{R}_+$ given by $g(x) = e^x$ is bijective. The inverse is $g^{-1} : \mathbb{R}_+ \to \mathbb{R}$ given by $g^{-1}(x) = \ln x$.

Definition 4. We say that two sets X and Y have *the same cardinality* if there exists a bijection $f : X \to Y$.

It is obvious that two finite sets X and Y have the same cardinality if and only if X and Y have the same number of elements.

Definition 5. A set S is called *countable* if it is finite, or if it has the same cardinality as $\mathbb{N} = \{1, 2, \dots\}$ the set of the natural numbers.

Clearly $\mathbb{N}$ is countable. If x is an element, not in $\mathbb{N}$, then the union $\mathbb{N} \cup \{x\}$ has the same cardinality as $\mathbb{N}$. Indeed we can define a bijection $f : \mathbb{N} \cup \{x\} \to \mathbb{N}$

by $f(n) = n + 1$ for all $n \in \mathbb{N}$ and $f(x) = 1$. It is not hard to see that f is a bijection.

The set $\mathbb{N} \times \{1, 2\}$ which is the union of the two infinite sets $\{(n, 1) \mid n \in \mathbb{N}\}$ and $\{(n, 2) \mid n \in \mathbb{N}\}$ is again countable. We can define $g : \mathbb{N} \times \{1, 2\} \to \mathbb{N}$ by $g(n, 1) = 2n$ and $g(n, 2) = 2n - 1$ for all $n \in \mathbb{N}$. It is again not hard to see that g is a bijection.

Finally $\mathbb{N} \times \mathbb{N} = \{(n, m) \mid n, m \in \mathbb{N}\}$ is also countable. Indeed we define $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $h(n, m) = 2^{n-1}(2m - 1)$. One can check that h is bijective. Again h is bijective.

One can also show that a countable union of countable sets is again countable.

There are sets which are not countable. For example $\mathbb{R}$ the set of the real numbers. To see this, we use Cantor's diagonal argument: Suppose that $f : \mathbb{N} \to \mathbb{R}$ is a bijection. Let us write $f(j)$ in its infinite decimal expansion:

$$\cdots .a_{j,1}a_{j,2}a_{j,3}a_{j4}\cdots$$

where $a_{j,1}, a_{j,2}, \ldots$ are digits in the set $\{0, 1, 2, \ldots, 9\}$ and there also may be a finite number of digits before the decimal point. Consider the real number β defined by

$$\beta = 0.b_1b_2b_3\cdots$$

where $b_i = 0$ if $a_{i,i} > 0$ and $b_i = 1$ if $a_{i,i} = 0$. Assuming that f was a bijection, there must be a k such that $f(k) = \beta$. This means that $a_{k,k} = \beta_k$ which contradicts the definition of β. This shows that there cannot be such a bijection f.

Theorem 1. *Suppose that $f : X \to Y$ and $g : Y \to X$ are injective. Then there exists a bijection between X and Y.*

Proof. Let $X_1 = X \setminus g(Y)$ and $Y_1 = Y \setminus f(X)$. Define inductively $X_n = g(Y_{n-1})$ and $Y_n = f(X_{n-1})$. By induction on n and m one shows that X_n and X_{n+m} are disjoint and also Y_n and Y_{n+m} are disjoint. We define $h : X \to Y$ by $h(x) = f(x)$ if $x \in X_n$ with n odd or $x \notin X_n$ for all n and $h(x) = y$ with $g(y) = x$ if $x \in X_n$ with n even. One can show that h is then a bijection. □

The previous theorem can come in handy. For example one can show that every subset of a countable set is again countable as follows. If X is a subset of $\mathbb{N}$, then either X is finite or we can easily construct an injective map $f : \mathbb{N} \to X$ (for example we can define $f(n)$ as the smallest element of $X \setminus \{f(1), f(2), \ldots, f(n-1)\}$ for all n).

In the latter case, we have injective maps $X \to \mathbb{N}$ and $\mathbb{N} \to X$ and therefore X and $\mathbb{N}$ must have the same cardinality.

Also the set of rational numbers, $\mathbb{Q}$ is countable. It suffices to show that the set $\mathbb{Q}$ of positive rational numbers is countable (why?). We can define an injective map $f : \mathbb{N} \to \mathbb{Q}$ by $f(n) = n$. Also we can define an injective map $g : \mathbb{Q} \to \mathbb{N}$ by $g(a/b) = 2^b(6a - 1)^2$ whenever the greatest common divisor of a and b is equal to

1 and b is positive (check that g is indeed well-defined and injective). This shows that $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality.

PROBLEMS

Problem 1. * Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by a polynomial of even degree. Show that f is not injective and also not surjective.

Problem 2. * If $f : X \to Y$ and $g : Y \to Z$ are functions, then the composition $g \circ f : X \to Z$ is defined by

$$g \circ f(x) = g(f(x)).$$

Use the definitions to prove the following results.

(a) If f and g are injective, then $g \circ f$ is also injective.
(b) If f and g are surjective, then $g \circ f$ is also surjective.
(c) If f and g are bijective, then $g \circ f$ is also bijective.

Problem 3. ** Let X be a set of n elements and Y be a set of m elements.

(a) Give a formula for the number of injective functions $f : X \to Y$.
(b) Give a formula fo the number of surjective functions $f : X \to Y$ in the case Y has $m = 3$ elements (can you generalize this to arbitrary m?).

Problem 4. *** Suppose that S is a set of intervals of the form (a, b) with $a, b \in \mathbb{R}$ and $a < b$. If S is not countable, then there must be two distinct elements of S, say (a, b) and (a', b') such that $(a, b) \cap (a', b') \neq \emptyset$. (*Hint:* First show that every such interval contains a rational number.)

Problem 5. ** Show that $\mathbb{Z}$ (the integers) is countable by giving an explicit bijection with $\mathbb{N}$.

Problem 6. **** Let T_n be the set sequences $(a_1, a_2, \dots, a_n)$ with $a_1, a_2, \dots, a_n \in \mathbb{N}$. Let T be the union of $T_1, T_2, T_3, \dots$. Prove that T is again countable by giving an *explicit* bijection between T and $\mathbb{N}$. (*Hint:* unique factorization into prime numbers.)

Problem 7. *** Show that $\mathbb{R}$ and $\mathbb{R}^2$ have the same cardinality. (*Hint:* It is easy to find an injective map $\mathbb{R} \to \mathbb{R}^2$. Now we need to find (not necessarily continuous) injective map $\mathbb{R}^2 \to \mathbb{R}$. Interlace the decimal expansion of the two coordinates of $\mathbb{R}^2$ to get a single real number.)

Problem 8. **** Let $\mathbb{Q}_+$ be the set of the positive rational numbers. We define $f : \mathbb{N} \to \mathbb{Q}_+$ inductively as follows. $f(1) = 1$, $f(2n) = f(n) + 1$, $f(2n + 1) = 1/f(2n)$ for all $n \in \mathbb{N}$. Show that f is a bijection.

Problem 9. ** Show that the set of all functions $f : \mathbb{N} \to \mathbb{N}$ is not countable by using a similar argument as Cantor's.

Problem 10. *** If X is a set, then there does not exists a surjective function $f : X \to \mathcal{P}(X)$ where $\mathcal{P}(X)$ is the set of all subsets of X. (*Hint:* Use something similar to Cantor's diagonal argument.)

Problem 11. ***** An increasing function $f : \mathbb{R} \to \mathbb{R}$ is always continuous at all but countably many points.