MATH 289, PROBLEM SET 1
MATHEMATICAL INDUCTION
DUE 1/14/2004

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1. In(tro)duction

We will denote the set of integers by

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

This problem set is about a fundamental property of the set of *positive integers*

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}.$$

Beware, some authors may use $\mathbb{N}$ for the nonnegative integers.

Suppose that we want to prove that

"$P(n)$ is true for every positive integer n",

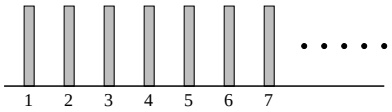
where $P(n)$ is a proposition (statement) which depends on a positive integer n. Proving $P(1)$, $P(2)$, $P(3)$, etc., would take an infinite amount of time. Instead we can use the so-called *induction principle*.

Axiom 1. *Assume that k is an integer and for each $n \geq k$, $P(k)$ is a proposition.*

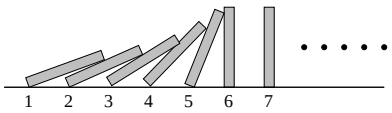
(1) *Suppose that $P(k)$ is true, and*
(2) *for any integer $m \geq k$ for which $P(m)$ is true, $P(m+1)$ is true.*

Then $P(n)$ is true for all integers $n \geq k$.

The induction principle is often compared to an infinite sequence of dominos tiles, numbered 1,2,3, etc.



If the m-th domino tile falls, it will hit the $(m+1)$-th domino tile and the $(m+1)$-th domino tile will fall as well. If the first domino tile falls, then *all* domino tiles will fall down. (Here $P(n)$ is the statement:"the n-th domino tile falls down")



Date: Winter 2004.

Since the induction principle is intuitively clear, we will simply accept it without proof. This is why it is called an axiom. (We cannot formally prove the induction principle without making other, similar assumptions.)

A typical example of the induction principle is the following:

Example 1. Prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \tag{1}$$

for every positive integer n.

Discussion. Since (1) depends on a positive integer n, it is natural to try mathematical induction. Equation (1) is true for $n = 1$. Let us assume that (1) is true for $n = m$:

$$1 + 2 + \cdots + m = \frac{m(m+1)}{2}. \tag{2}$$

Then we have to prove that

$$1 + 2 + \cdots + m + (m+1) = \frac{(m+1)(m+2)}{2}. \tag{3}$$

We recognize that the left-hand side of (2) forms a substantial part of the left-hand side of (3). If we assume (2), then (3) is equivalent to

$$\frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2}.$$

This is easy to check. We now write down a formal proof.

Proof. We prove (1) by induction on n. For $n = 1$ we check that

$$1 = \frac{1 \cdot (1+1)}{2}.$$

Suppose that (1) is true for $n = m$. Then

$$1 + 2 + \cdots + m + (m+1) = (1 + 2 + \cdots + m) + (m+1) =$$

$$= \frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2}.$$

so (1) is true for $n = m + 1$. Now (1) is true for all positive integers n by the induction principle. ☺

Remark 1. When the German mathematician Carl Friedrich Gauss (1777–1855) was 10 years old, his school teacher gave the class an assignment to add all the numbers from 1 to 100. Gauss gave the answer almost immediately: 5050. This

is how (we think) he did it: Write the numbers from 1 to 100 from left to right. Write under that the numbers from 1 to 100 in reverse order.

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & 100 \\ 100 & 99 & 98 & \cdots & 1 \\ \hline \underbrace{101 \quad 101 \quad 101 \quad \cdots \quad 101}_{100} \end{array}$$

Each of the 100 column sums is 101. This shows that

$$2 \cdot (1 + 2 + \cdots + 100) = 100 \cdot 101$$

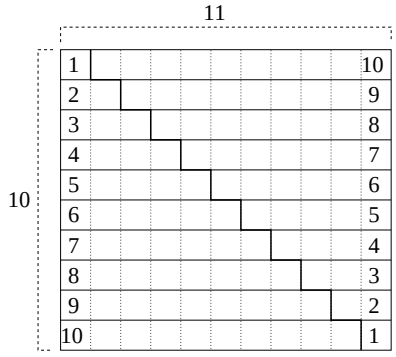
and

$$1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2} = 50 \cdot 101 = 5050.$$

This easily generalizes to a proof of (1). Gauss' proof can be graphically presented. For example, to see that

$$2 \cdot (1 + 2 + \cdots + 10) = 10 \cdot 11,$$

look at the following picture:



Remark 2. One of the earliest uses of mathematical induction appears in the work of French mathematician Blaise Pascal (1623–1662). Some sources attribute the invention of the *roulette wheel* to Pascal (while he was attempting to create a perpetual motion machine). (Another theory states that the roulette wheel was brought to Europe from China.) The modern roulette wheel has the numbers 0 to 36. The sum of all the numbers on a roulette wheel is

$$1 + 2 + \cdots + 36 = \frac{36 \cdot 37}{2} = 666.$$

This is the biblical "mark of the beast". Coincidence?

A formula similar to (1) exists for the sums of squares, namely

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{4}$$

Example 2. Give and prove a formula for

$$1^3 + 2^3 + \cdots + n^3$$

Discussion. What would be the *form* of such a formula? We have seen similar examples, namely (1) and (4). We can also add the formula

$$1^0 + 2^0 + 3^0 + \cdots + n^0 = n.$$

Let

$$p_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$$

where $k \in \mathbb{N}$. The examples so far suggest that $p_k(n)$ is a polynomial of degree $k+1$ (and that the leading coefficient is $\frac{1}{k+1}$). Let us *assume* that

$$p_3(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

is a polynomial of degree 4. Since $p_3(0)$ is an empty sum, we have that $p_3(0) = 0$. We can write

$$1^3 + 2^3 + \cdots + n^3 = an^4 + bn^3 + cn^2 + dn \tag{5}$$

for certain real numbers a, b, c, d. Suppose that for some fixed values of a, b, c, d, (5) is true for all $n \in \mathbb{N}$. How would we prove such a formula? Of course we use induction. Let $Q(n)$ be the proposition "(5) is true". We know that $Q(0)$ is true. We want to show that $Q(m)$ implies $Q(m+1)$, or equivalently that $Q(m-1)$ implies $Q(m)$. So

$$1^3 + 2^3 + \cdots + (m-1)^3 = a(m-1)^4 + b(m-1)^3 + c(m-1)^2 + d(m-1) \tag{6}$$

should imply

$$1^3 + 2^3 + \cdots + (m-1)^3 + m^3 = am^4 + bm^3 + cm^2 + dm. \tag{7}$$

The left-hand sides of (6) and (7) look very similar. If we subtract (6) from (7) we obtain

(8)

$$\begin{aligned} m^3 &= a(m^4-(m-1)^4)+b(m^3-(m-1)^3)+c(m^2-(m-1)^2)+d(m-(m-1)) = \\ &= a(4m^3-6m^2+4m-1)+b(3m^2-3m+1)+c(2m-1)+d = \\ &= m^3(4a)+m^2(-6a+3b)+m(4a-3b+2c)+(-a+b-c+d) \end{aligned}$$

If (8) is true for all m, then (6) indeed would imply (7). Perhaps we can choose a, b, c, d in such way that (8) always holds. Comparing coefficients in (8) gives us the linear equations:

$$1 = 4a \tag{9}$$

$$0 = -6a + 3b \tag{10}$$

$$0 = 4a - 3b + 2c \tag{11}$$

$$0 = -a + b - c + d \tag{12}$$

We solve the system of equations. We find $a = \frac{1}{4}$. Then we get $3b = 6a = \frac{3}{2}$, so $b = \frac{1}{2}$. Then we have $2c = -4a + 3b = -1 + \frac{3}{2} = \frac{1}{2}$ so $c = \frac{1}{4}$. Finally we get $d = a - b + c = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0$. We now should conjecture the following formula:

$$1^3 + 2^3 + \cdots + n^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2.$$

Finding this formula was the hard part. It is now not so hard to prove this formula by induction. If we write down the proof, we do not really have to explain how we obtained the formula. We just "drop the formula out of the sky" and give a (short) proof by induction. Here we go:

Proof. We will prove that

$$1^3 + 2^3 + \cdots + n^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2. \tag{13}$$

by induction on n. The case $n = 0$ is clear, because both sides of the equation are equal to 0. If (13) is true for $n = m - 1$, then

$$1^3 + 2^3 + \cdots + (m-1)^3 = \tfrac{1}{4}(m-1)^4 + \tfrac{1}{2}(m-1)^3 + \tfrac{1}{4}(m-1)^2.$$

From this follows that

$$1^3 + 2^3 + \cdots + (m-1)^3 + m^3 = \tfrac{1}{4}(m-1)^4 + \tfrac{1}{2}(m-1)^3 + \tfrac{1}{4}(m-1)^2 + m^3 =$$
$$= \tfrac{1}{4}(m^4 - 4m^3 + 6m^2 - 4m + 1) + \tfrac{1}{2}(m^3 - 3m^2 + 3m - 1) + \tfrac{1}{4}(m^2 - 2m + 1) + m^3 =$$
$$= \tfrac{1}{4}m^4 + \tfrac{1}{2}m^3 + \tfrac{1}{4}m^2,$$

so (13) is true for $n = m$. By induction follows that (13) is true for all $n \in \mathbb{N}$. ☺

Notice that

$$\tfrac{1}{4}m^4 + \tfrac{1}{2}m^3 + \tfrac{1}{4}m^2 = (\tfrac{1}{2}n(n+1))^2$$

which leads to the following estetic formula:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

Example 3. What is the value of

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots ?$$

Discussion. Let us compute the partial sums. Perhaps we will find a pattern.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{2}{6} + \frac{4}{6} = \frac{2}{3},$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{2}{3} + \frac{1}{12} = \frac{8}{12} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4},$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{3}{4} + \frac{1}{20} = \frac{15}{20} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5}.$$

A pattern emerges. Namely, it seems that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}. \tag{14}$$

The form of the formula suggests that we can prove it by induction. Besides checking (14) for $n = 1$, we have to prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{m(m+1)}=1-\frac{1}{m+1} \tag{15}$$

implies

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(m+1)(m+2)}=1-\frac{1}{m+2}. \tag{16}$$

Let us assume (15). We recognize that the left-hand side of (15) forms a substantial part of the left-hand side of (16). We can use this:

$$\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{m(m+1)}\right)+\frac{1}{(m+1)(m+2)}=$$

$$\big(1-\frac{1}{m+1}\big)+\frac{1}{(m+1)(m+2)}.$$

Now (16) follows because

$$\frac{1}{(m+1)(m+2)}=\frac{1}{m+1}-\frac{1}{m+2}.$$

After proving (14) we can take the limit $n\to\infty$ to find the the series converges to 1. We are ready to write down a formal proof now.

Proof. By induction on n we prove:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}. \tag{17}$$

For $n = 1$ we check

$$\frac{1}{1\cdot 2}=1-\frac{1}{2}.$$

If (17) is true for $n = m$, then

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{m(m+1)}+\frac{1}{(m+1)(m+2)}=$$

$$=\big(1-\frac{1}{m+1}\big)+(\frac{1}{m+1}-\frac{1}{m+2})=1-\frac{1}{m+2}.$$

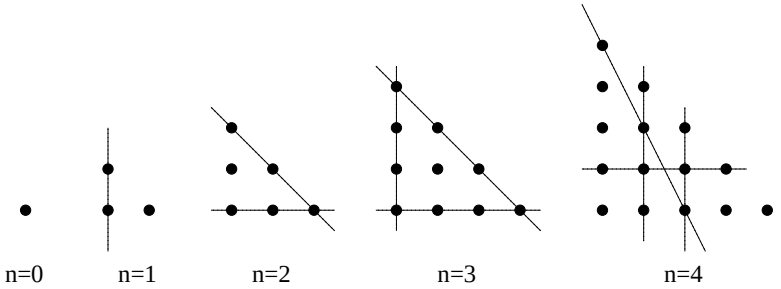
Hence (17) is true for $n = m+1$. By induction, (17) is true for all integers $n \geq 1$. We have

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots=\lim_{n\to\infty}\big(1-\frac{1}{n+1}\big)=1.$$

☺

Example 4 (UMUMC, 1988)**.** Let S_n be the set of all pairs (x, y) with integral coordinates such that $x \geq 0$, $y \geq 0$ and $x + y \leq n$. Show that S_n cannot be covered by the union of n straight lines.

Discussion. First we should try a few small cases, say $n = 0, 1, 2, 3, 4$:



How can we use induction here? Notice that S_n is a subset of S_{n+1}. Suppose that we already have proven that one needs at least $n+1$ lines to cover S_n. Obviously, one will need at least $n+1$ to cover S_{n+1} because S_n is a subset of S_{n+1}. Suppose that $n+1$ lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$ cover S_n. Could these lines in fact cover the whole of S_{n+1}? We should concentrate on $C := S_{n+1} \setminus S_n$, the complement of S_n in S_{n+1}. We have

$$C_{n+1} = \{(0, n+1), (1, n), \ldots, (n+1, 0)\}.$$

This is a set of $n+2$ points lying on the line $x + y = n + 1$. Is it possible that $\ell_1, \ell_2, \ldots, \ell_{n+1}$ cover C_{n+1}? For every i we have that $\ell_i \cap C_{n+1}$ consists of at most 1 point unless ℓ_i is equal to the line $x + y = n + 1$. If none of the lines ℓ_i is equal to the line $x + y = n + 1$ then $\ell_1, \ell_2, \ldots, \ell_{n+1}$ cover at most $n+1$ of the $n+2$ points of C_{n+1}. So $\ell_1, \ldots, \ell_{n+1}$ do not cover S_{n+1}. If ℓ_i is the line $x + y = n + 1$ for some i then $\ell_1, \ell_2, \ldots \widehat{\ell_i}, \ldots, \ell_{n+1}$ (we omit ℓ_i) cover S_n. This contradicts the assumption that we need at least $n+1$ lines to cover S_n. We conclude that we need at least $n+2$ lines to cover S_{n+1}.

We now write down the proof. Notice that it is not really necessary to use a proof by contradiction. A proof is usually more elegant if one avoids reasoning by contradiction.

Proof. We prove the statement by induction on n, the case $n = 0$ being trivial. Suppose that one needs at least $n+1$ lines to cover S_n. Define $C_{n+1} = S_{n+1} \setminus S_n$. The set C_{n+1} consists of $n+2$ points on the line $x + y = n + 1$. Suppose that k lines $\ell_1, \ell_2, \ldots, \ell_k$ cover S_{n+1}.

case 1: One of the lines is equal to the line $x + y = n + 1$. Without loss of generality we may assume that ℓ_k is equal to the line $x + y = n + 1$. Then $\ell_1, \ell_2, \ldots, \ell_{k-1}$ cover S_n because $\ell_k \cap S_n = \emptyset$. From the induction hypothesis follows that $k - 1 \geq n + 1$, so $k \geq n + 2$.

case 2: None of the lines are equal to the line $x + y = n + 1$. Then each of the lines intersects the line $x + y = n + 1$ in at most one point, and therefore it intersects the set C_{n+1} in at most one point. Since C_{n+1} has $n+2$ elements, there must be at least $n+2$ lines.

So in both cases we conclude that one needs at least $n+2$ lines to cover S_{n+1}. ☺

Problem 1. *

(a) Prove that

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

for every real number x and every positive integer n.

(b) If x is a real number with $|x| < 1$ then

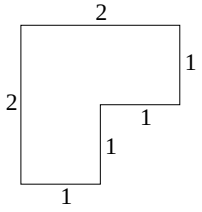
$$1 + x + x^2 + \cdots = \frac{1}{1 - x}.$$

Problem 2. ** Show that

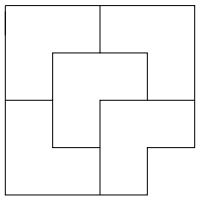
$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

for all $n \in \mathbb{N}$.

Problem 3. *** Cut out a 1×1 corner of a $2^n \times 2^n$ chess board ($n \geq 1$). Show that the remainder of the chess board can be covered with L-shaped tiles (see picture).



The case $n = 2$ is shown below.



Problem 4. *** Suppose that we have n lines in the plane such that (i) no two lines are parallel and, (ii) no three lines go through 1 point. In how many regions do these lines divide the plane? Prove your formula.

2. Strong Induction

The following example illustrates that sometimes one has to make a statement stronger in order to be able to prove it by induction.

Example 5. Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{999,999}{1,000,000} < \frac{1}{1000}.$$

Discussion. Since $1000 = \sqrt{1,000,000}$ one might suggest that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n}} \tag{18}$$

for all $n \geq 1$. Let us try to prove (18). We can check (18) for small n (which gives some validity to our conjecture that this inequality holds). Suppose that (18) holds for $n = m$:

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2m-1}{2m} < \frac{1}{\sqrt{2m}} \tag{19}$$

We have to prove (18) for $n = m + 1$:

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2m+1}{2m+2} < \frac{1}{\sqrt{2m+2}}. \tag{20}$$

If we divide (20) by (19) we obtain

$$\frac{2m+1}{2m+2} \leq \sqrt{\frac{2m}{2m+2}}. \tag{21}$$

If (19) and (21) are true, then (20) is true. By squaring (21) we see that (21) is equivalent to

$$\left(\frac{2m+1}{2m+2}\right)^2 \leq \frac{2m}{2m+2}$$

and to

$$(2m+1)^2 \leq (2m+2)(2m) \tag{22}$$

So if (22) is true then our induction proof is complete. Unfortunately (22) is not true and we are stuck. ☹

Apparently we cannot easily prove (18) by induction. Sometimes it is easier to prove a *stronger* statement by induction. We want to prove an inequality of the form

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{f(n)} \tag{23}$$

for some function $f(n)$ by induction on n. We want $f(500,000) > 1000$ so that (23) would imply the original problem. (Perhaps we want also that $f(n) > \sqrt{2n}$ for all n so that (23) implies (18)). For the induction to work, we need that $f(1) < 2$ and

$$\frac{2m+1}{2m+2} \leq \frac{f(m)}{f(m+1)}$$

for all $m \geq 1$. With some puzzling we see that $f(m) = \sqrt{2m+1}$ works. This means that the inequality

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}} \tag{24}$$

can easily be proven by induction.

Proof. We prove

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}} \tag{25}$$

by induction on n. The case $n = 1$ is clear because

$$\frac{1}{2} < \frac{1}{\sqrt{3}}.$$

Suppose that (25) is true for $n = m$:

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2m-1}{2m} < \frac{1}{\sqrt{2m+1}} \tag{26}$$

Since

$$(2m+1)(2m+3) = (2m+2)^2 - 1 < (2m+2)^2$$

we have that

$$\left(\frac{2m+1}{2m+2}\right)^2 < \frac{2m+1}{2m+3}$$

and

$$\frac{2m+1}{2m+2} < \sqrt{\frac{2m+1}{2m+3}}. \tag{27}$$

Multiplying (26) by (27) yields

$$\frac{1}{2}\frac{3}{4}\cdots\frac{2m+1}{2m+2} < \frac{1}{\sqrt{2m+3}}, \tag{28}$$

so (25) is true for $n = m+1$. This shows that (25) is true for all positive integers n. In particular, for $n = 500,000$ we get

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{999,999}{1,000,000} < \frac{1}{\sqrt{1,000,001}} < \frac{1}{1000}.$$

☺

What we did in the previous example is the following. Instead of proving a statement $P(n)$ for all $n \in \mathbb{N}$, we formulated a stronger statement $Q(n)$, such that $Q(n)$ implies $P(n)$ for any given positive integer n. We chose $Q(n)$ such that $Q(n)$ was easy to prove by induction on n.

Below is a trickier proof of Example 5.

Proof. Let

$$A = \frac{1\cdot 3\cdot 5\cdots 999,999}{2\cdot 4\cdot 6\cdots 1,000,000}$$

and

$$B = \frac{2\cdot 4\cdot 6\cdots 1,000,000}{3\cdot 5\cdot 7\cdots 1,000,001}.$$

Clearly $A < B$ because

$$\frac{1}{2} < \frac{2}{3}, \frac{3}{4} < \frac{4}{5}, \ldots, \frac{999,999}{1,000,000} < \frac{1,000,000}{1,000,001}.$$

It follows that

$$A^2 < AB = \frac{1}{1,000,001} < \frac{1}{1,000,000}$$

and $A < 1000^{-1}$. ☺

Example 6. Prove that every integer $n \geq 2$ is a product of prime numbers.

Discussion. Recall that a positive integer p is a prime number if it has exactly two positive divisors, namely 1 and p itself. For example, $2, 3, 5, 7, 11, 13$ are prime numbers. Other numbers are products of primes, for example $4 = 2 \cdot 2$, $6 = 2 \cdot 3$, $8 = 2 \cdot 2 \cdot 2$, $9 = 3 \cdot 3$, $10 = 2 \cdot 5$ and $12 = 2 \cdot 2 \cdot 3$. The statement seems plausible. Perhaps we can prove the statement by induction. Let $P(n)$ be the statement: "n is a product of prime numbers". Clearly $P(2)$ is true. We have to prove that $P(m)$ implies $P(m+1)$. It seems that we are in a dead end. It is not clear at all how the fact that m is a product of prime numbers can help us prove that $m+1$ is a product of prime numbers. ☹

How can we prove that $m+1$ is a product of prime numbers? If $m+1$ were a prime number itself, then we are done. Otherwise $m+1$ will have a nontrivial decomposition $m+1 = ab$ with $a, b \leq m$. If we would know that all integers $\leq m$ are products of primes, then we would know that $m+1$ is a product of primes.

It therefore seems natural to prove a slightly different statement by induction. Let $Q(n)$ be the statement:"every integer r with $2 \leq r \leq n$ is a product of prime numbers". The statement $Q(n)$ can now be proven by induction without too much effort. Instead of proving $P(n)$ by induction, we prove the statement "$P(r)$ is true for all $r \leq n$" by induction. This idea is called *strong induction*. Here is the proof.

Proof. Let $Q(n)$ be the statement:"every integer r with $2 \leq r \leq n$ is a product of prime numbers." We use induction on n to prove that $Q(n)$ holds for all integers $n \geq 2$.

For $n = 2$ the statement is true because 2 is a prime number. Suppose that $Q(m)$ is true. We will prove $Q(m+1)$. Suppose that $2 \leq r \leq m+1$. If $r \leq m$ then r is a product of prime numbers because $Q(m)$ is true. Suppose that $r = m+1$. If $m+1$ is a prime number, then $m+1$ is a product of prime numbers and we are done. Otherwise, $m+1$ can be written as a product ab with $1 \leq a, b \leq m$. Because $Q(m)$ is true, both a and b are products of prime numbers. Hence $m+1 = ab$ is a product of prime numbers.

We have shown that $Q(n)$ holds for all $n \geq 2$. In particular, every integer $r \geq 2$ is a product of prime numbers because $Q(r)$ is true. ☺

Example 7. Use the induction principle to show that every nonempty subset of $\mathbb{N}$ has a smallest element.

Discussion. It is not immediately clear how to set up an induction here. There is no "n" on which we can base the induction. Suppose that S is a nonempty set. It contains at least one nonnegative integer n. Perhaps we could base the induction on this positive integer. So let $P(n)$ be the statement:"if S is a subset of $\mathbb{N}$ containing n then S must have a smallest element". Proving $P(n)$ by induction is perhaps possible. However, it is easier to use strong induction in this example. The proof then goes without any difficulties.

Proof. Let $Q(n)$ be the statement:"If S is a subset of $\mathbb{N}$ containing an integer r with $r \leq n$, then S has a smallest element." We are going to prove $Q(n)$ by induction on n. The case $n = 1$ is clear. Any set S containing 1 has a smallest element, namely 1. Suppose that $Q(m)$ is true. We will prove $Q(m+1)$. Let S be a set containing an element r with $r \leq m+1$. If S contains an element s with $s \leq m$ then S has a smallest element because $Q(m)$ is true. Otherwise, every element of S is $\geq m+1$. We must have $r = m+1$ and r must be the smallest element of S. So in any case, S has a smallest element. ☺

Problem 5. *** Suppose that we have an $m \times n$ chocolate bar. We break the chocolate bar into two pieces. Then we take one piece and break it into two. We keep repeating this until we are only left with mn pieces of size 1×1. How many times do we have to break the chocolate bar? Prove your formula. (In particular, show that the number of breaks needed does not depend on how you proceed.)

Problem 6. *** Suppose that $A_1, A_2, A_3, \ldots$ is a sequence of positive integers such that $A_1 = 1$ and $A_i < A_{i+1} \leq 2A_i$ for all positive integers i. Prove that every positive integer n can be written as a sum of distinct A_i's.

3. Induction in Definitions

We can also use induction in a definition. For example, the Fibonacci numbers is a sequence of numbers $F_0, F_1, F_2, \ldots$ defined by $F_0 = F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1.$$

By (strong) induction on n we can prove that F_n is well-defined for all integers $n \geq 0$. The first few Fibonacci numbers are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

The sum notation is an example of a recursive definition. Suppose that $f(n)$ is some function. If a, b are integers and $a \leq b+1$ then we define

$$\sum_{n=a}^{b} f(n)$$

as follows.

$$\sum_{n=a}^{a-1} f(n) = 0$$

and

$$\sum_{n=a}^{b} f(n) = f(b) + \sum_{n=a}^{b-1} f(n) \tag{29}$$

if $b \geq a$.

One can then formally prove by induction that

$$\sum_{n=a}^{c} f(n) = \sum_{n=a}^{b} f(n) + \sum_{n=b+1}^{c} f(n).$$

if $a, b, c \in \mathbb{Z}$ and $a - 1 \leq b \leq c$. (Induction on c. Start with $c = b$.)

Similarly we have the product notation.

$$\prod_{n=a}^{a-1} f(n) = 1$$

and

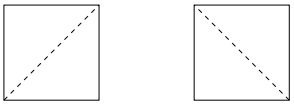
$$\prod_{n=a}^{b} f(n) = f(b) \prod_{n=a}^{b-1} f(n).$$

if $b \geq a$.

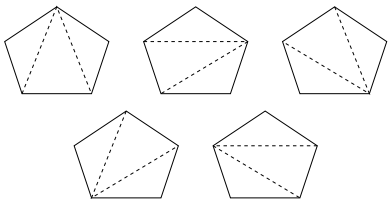
Example 8. A triangulation of a convex n-gon is a partition of the area of the n-gon into triangles such that the vertices of each triangle is a vertex of the n-gon. How many distinct triangulations does a convex 10-gon have?

Discussion. Let A_n be the number of triangulations of an n-gon. Let us find the value of A_n for small n.

We have $A_3 = 1$ and $A_4 = 2$:

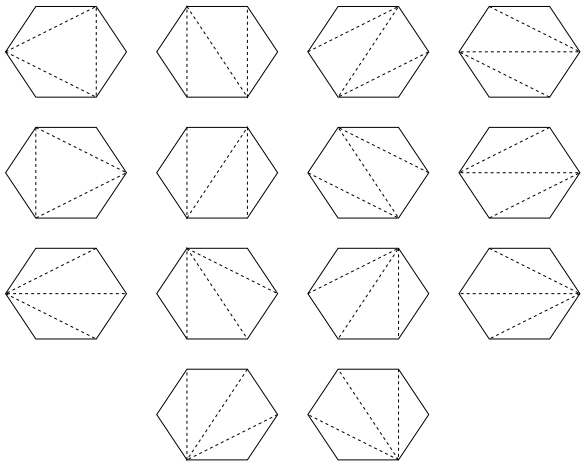


$A_5 = 5$:



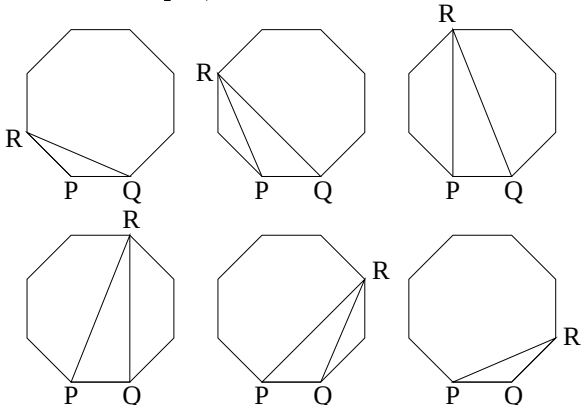
I

$A_6 = 14$:



It becomes more and more clear that it may not be feasible to write down all triangulations of an 10-gon. As n gets larger, we need a more systematic way of counting the possibilities to make sure that we are not forgetting any case.

Let P and Q be two fixed adjacent vertices of the n-gon. For each triangulation, there is a unique vertex R of the n-gon ($R \neq P, Q$) such that PQR is a triangle in the triangulation. For example, for $n = 8$ there are the following cases:



For fixed R, the complement of the triangle PQR within the n-gon is a union of an m-gon and a $(n+1-m)$-gon. The m-gon has A_m triangulations, and the $(n+1-m)$-gon has A_{n+1-m} triangulations. This gives $A_m A_{n+1-m}$ triangulations for this particular choice of R. From this we see the equation:

$$A_n = A_2 A_{n-1} + A_3 A_{n-2} + \cdots + A_{n-1} A_2.$$

where we define $A_2 = 1$. In particular,

$$\begin{aligned}
A_7 &= 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42 \\
A_8 &= 1 \cdot 42 + 1 \cdot 14 + 2 \cdot 5 + 5 \cdot 2 + 14 \cdot 1 + 42 \cdot 1 = 132 \\
A_9 &= 1 \cdot 132 + 1 \cdot 42 + 2 \cdot 14 + 5 \cdot 5 + 14 \cdot 2 + 42 \cdot 1 + 132 \cdot 1 = 429 \\
A_{10} &= 1 \cdot 429 + 1 \cdot 132 + 2 \cdot 42 + 5 \cdot 14 + 14 \cdot 5 + 42 \cdot 2 + 132 \cdot 1 + 429 \cdot 1 = 1430.
\end{aligned}$$

By the way, if one defines $C_n = A_{n+2}$ for all $n \geq 2$, then C_n are the so-called *Catalan numbers*. The Catalan numbers have many interesting interpretations (which we will not discuss now). It is known that

$$C_n = \frac{\binom{2n}{n}}{n+1}.$$

So for example $A_{10} = C_8 = \binom{16}{8}/9 = 1430$.

Problem 7. ** Show that the sum of the squares of two consecutive Fibonacci numbers is again a Fibonacci number.

Problem 8. **** [Putnam 1985] Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx}(f_{n+1}(x)) = (n+1)f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Problem 9. *** [Putnam 1990] Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are,

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

Problem 10. *** [Putnam 1993] Let $\{x_n\}_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

4. Extra Problems

Problem 11. * Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

Problem 12. ** Find and prove a formula for

$$1^4 + 2^4 + \cdots + n^4.$$

Problem 13. ******* Suppose that d and m are nonnegative integers. Prove that

$$\sum_{n=0}^{m} n^d = \sum_{j=0}^{d} \left\{ \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} i^d \right\} \binom{m+1}{j+1}.$$

Problem 14. **** [Putnam 1985] Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

Problem 15. * Show that the Fibonacci numbers satisfy

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}$$

for all positive integers n.