PROBLEM SET 2: RECURRENCE RELATIONS (DUE 1/21/2004)

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A sequence of numbers $a_0, a_1, a_2, \dots$ is said to satisfy a recurrence relation if a_n can be expressed in terms of a_i with $i < n$. One important example are the Fibonacci numbers. The Fibonacci numbers are defined by:

$$F_0 = 1,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1} \text{ for all } n \geq 1.$$

So we obtain $F_2 = 1 + 1 = 2, F_3 = 2 + 1, F_4 = 3 + 2 = 5, F_5 = 5 + 3 = 8$ and so forth. The beginning of the Fibonacci sequence is:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, \dots$$

We can find a closed formula for Fibonacci numbers as follows. Consider the recurrence relation

$$F_{n+1} = F_n + F_{n-1}$$

without the initial condition. Suppose it has a solution of the form $F_n = \lambda^n$ with $\lambda \neq 0$. This implies that

$$\lambda^{n+1} = \lambda^n + \lambda^{n-1}$$

which is equivalent to

$$\lambda^2 = \lambda + 1.$$

Solving

$$\lambda^2 - \lambda - 1 = 0$$

gives us two solutions

$$\frac{1 \pm \sqrt{5}}{2}$$

which we will call $\lambda_1 = (1 - \sqrt{5})/2$ and $\lambda_2 = (1 + \sqrt{5})/2$. We have that $F_n = \lambda_1^n$ and $F_n = \lambda_2^n$ are solutions of the recurrence relation. Also, it is easy to check that $F_n = a\lambda_1^n + b\lambda_2^n$ is also a solution (this is called the *superposition principle*). A sum of two solutions is again a solution. If you multiply a solution with a constant, then this is again a solution. In other words the set of solutions is a vector space over $\mathbb{R}$. To find a solution with the initial conditions $F_0 = 1$ and $F_1 = 1$, we solve for a and b:

$$1 = F_0 = a\lambda_1^0 + b\lambda_2^0 = a + b$$

and

$$1 = F_1 = a\lambda_1 + b\lambda_2 = a\frac{1 - \sqrt{5}}{2} + b\frac{1 + \sqrt{5}}{2}.$$

If we substitute $a = 1 - b$ in the second equation we find

$$b = \frac{\sqrt{5}+1}{2\sqrt{5}} = \frac{\lambda_2}{\sqrt{5}},\ a = \frac{1-\sqrt{5}}{2\sqrt{5}} = \frac{-\lambda_1}{\sqrt{5}}.$$

So we have

$$F_n = \frac{\lambda_2^{n+1} - \lambda_1^{n+1}}{\sqrt{5}}$$

Another way of looking at Fibonacci numbers is to look at the generating function. We define

$$F(x) = F_0 + F_1 x + F_2 x^2 + \cdots$$

which is called the generating function of the sequence $F_0, F_1, F_2, \ldots$. The conditions $F_0 = 1, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ show that

$$(1 - x - x^2)F(x) = 1.$$

This means that $F(x) = \frac{1}{1-x-x^2}$. In other words, the Fibonacci numbers are the coefficients of the power series of the rational function $1/(1 - x - x^2)$.

PROBLEMS

Problem 1. * Give a formula for the sequence A_n with $A_0 = 1, A_1 = 2$, $A_{n+1} = A_n + 6A_{n-1}$.

Problem 2. ** Give a formula for the sequence S_n with $S_0 = 0$, $S_1 = 1$, $S_2 = 4$, $S_{n+1} = 3S_n - 3S_{n-1} + S_{n-2}$. (*Hint:* For this problem, compute a few numbers in the sequence, then guess.)

Problem 3. ** Consider the sequence D_n defined by $D_0 = 1$, $D_1 = \frac{3}{5}$, $D_{n+1} = \frac{6}{5}D_n - D_{n-1}$. Prove that $|D_n| \leq 1$ for all n.

Problem 4. *** Compute

$$\lim_{n\to\infty} (2+\sqrt{2})^n - \lfloor (2+\sqrt{2})^n \rfloor$$

where $\lfloor x \rfloor$ is the largest integer $\leq x$.

Problem 5. **** A famous sequence of numbers are the so-called Catalan numbers. They are defined by $C_0 = 1, C_1 = 1$ and

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

for $n \geq 1$. So we get the sequence

$$1, 1, 2, 5, 14, 42, \ldots.$$

Prove that the generating function $C(x)$ satisfies

$$C(x)^2 = \frac{C(x) - 1}{x}.$$

Find an explicit formula for $C(x)$ and prove that

$$C_n = \frac{\binom{2n}{n}}{n+1}.$$

Problem 6. **** Show that for every prime number p, there exists a Fibonnaci number F_n such that p divides F_n.

Extra Problems

Problem 7. *** The floor of a hallway has size 2×20. How many ways are there to tile the floor with tiles of size 1×2. (Hint: Let A_n be the number of ways to tile a $2 \times n$ floor. Deduce a recurrence relation for A_n.)

Problem 8. *** What is $F_0 + F_1/10 + F_2/10^2 + F_3/10^3 + \cdots$? In other words, add all the mumbers:

$$\begin{array}{ccccccccc}
1 & & & & & & & & \\
0 & . & 1 & & & & & & \\
0 & . & 0 & 2 & & & & & \\
0 & . & 0 & 0 & 3 & & & & \\
0 & . & 0 & 0 & 0 & 5 & & & \\
0 & . & 0 & 0 & 0 & 0 & 8 & & \\
0 & . & 0 & 0 & 0 & 0 & 1 & 3 & \\
0 & . & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Problem 9. *** Consider the sequence $A_0, A_1, A_2, \ldots$ defined by $A_0 = 10.1$ and $A_{n+1} = A_n^2 - 2$ for all $n \geq 0$. Find and prove a closed formula for A_n.

Problem 10. *** Consider the sequence of polynomials defined by $T_0(x) = 1$, $T_1 = x$ and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. Prove that

$$T_n(\cos(x)) = \cos(nx).$$

Also prove that

$$T_n(x) = \frac{(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n}{2}.$$