Linear algebra, test 1 answers, Sep 24 2018

In order to understand the later chapters, you need to know+understand 100% of this test. Read chapter 1 in the book and if you see something highlighted in a box, it means that you must know+memorize+understand it in order to understand the next chapters.

1. (15 points). Let

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} a \\ a \\ 1 \end{pmatrix}.$$

For which value(s) of a would w be an element of SPAN(v_1, v_2)?

Must know: Recall from section 1.3 (first box on page 30) that w is in SPAN(v_1, v_2), if and only if the system $x_1 v_1 + x_2 v_2 = w$ is consistent. Augmented matrix: $(v_1\ v_2\ \ w)$. Recall from section 1.2 (first box on page 21) that we can see if that system is consistent once we have computed a REF (a Row Echelon Form). Note: a REF is not unique (the RREF is unique) so your REF could look different and still be correct.

Answer: Row-reduce $(v_1\ v_2\ \ w)$ to obtain a REF. I did: $R_2 \leftarrow R_2 + R_1$, and then $R_3 \leftarrow R_2 + R_3$ to obtain

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & -1 & 2a \\ 0 & 0 & 1+2a \end{array}\right).$$

This is consistent if and only if $1 + 2a = 0$ (see page 21) so w is in the SPAN of v_1, v_2 if and only if $a = -1/2$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ send $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to $\begin{pmatrix} x_1 + x_2 \\ x_2 \\ 2x_1 + 3x_2 \end{pmatrix}$.

   (a) (15 points). Give the matrix of T.

   Must know: Theorem 10 on page 72.

Answer: $Te_1 = T\begin{pmatrix}1\\0\end{pmatrix}$ (i.e. $x_1 = 1$ and $x_2 = 0$) so $Te_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}$.

Likewise $Te_2 = T\begin{pmatrix}0\\1\end{pmatrix}$ (i.e. $x_1 = 0$ and $x_2 = 1$) so $Te_2 = \begin{pmatrix}1\\1\\3\end{pmatrix}$.

Then Theorem 10 tells us that the answer is $\begin{pmatrix}1 & 1\\0 & 1\\2 & 3\end{pmatrix}$.

(b) (5 points). Give the definition of injective (one-to-one).

Must know: second box on page 76.

One answer: $T : A \to B$ is *injective* if for every $b \in B$ there exists *at most* one $a \in A$ for which $T(a) = b$.

Note: there are many correct answers because there are many ways to reformulate this.
Reformulation 1: different elements of the domain go to different elements of the co-domain.
Reformulation 2: if $T(u) = T(v)$ then u must be v.

(c) (5 points). Give the definition of surjective (onto).

Must know: first box on page 76.

One answer: $T : A \to B$ is *surjective* if for every $b \in B$ there exists *at least* one $a \in A$ for which $T(a) = b$.

Note: there are many correct answers because there are many ways to reformulate this.
One reformulation: the range equals the co-domain.
Another: $T(x) = b$ has a solution for any b in the co-domain.

(d) (5 points). Is T injective?

Must know: Theorem 12 on page 78. In order to use the theorem, you need to combine it with results from prior sections. You need to know that the condition in Theorem 12 part (a) (columns span $\mathbb{R}^m$) is the same as saying "a pivot in all rows in a REF".
You also need to know that the condition in Theorem 12 part (b) (columns are linearly independent) is the same as saying "a pivot in all columns in a REF".

Answer: Yes.

(e) (5 points). Is T surjective?

Answer: No.

3. Let

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ -1 & 1 & -1 & 2 & 1 \\ 0 & -1 & -1 & -1 & -2 \end{pmatrix}.$$

(a) (10 points). Compute the rref (*reduced* row echelon form) of matrix A. Indicate which row-operations you used (e.g. $R_2 \leftarrow R_2 + R_1$).

Answer: I applied $R_2 \leftarrow R_2 + R_1$, $R_3 \leftarrow R_3 + R_2$, $R_2 \leftarrow R_2 - 2R_3$. It is OK if you applied different row-reductions, but the RREF is unique, so you must end up with this

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) (10 points). The matrix A is the augmented matrix of the following system of equations:

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Give *all* solutions for this system. Write the solutions in *parametric vector form.*

Note: don't do any more row-reduction for this question, you already did that in part (a).

Answer:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1-2x_3 \\ 2-x_3 \\ x_3 \\ 0 \end{pmatrix} \text{ Parametric form: } \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

(c) (10 points). Let $v_1, \ldots, v_5$ be the columns of A. Answer the following questions. Note: don't do any computation for this question, you already did that in part (a).

Must know: v_n is in the SPAN of $v_1, \ldots, v_{n-1}$ if and only if the augmented matrix $(v_1 \cdots v_n)$ is a consistent system. For each of the questions below, we have already computed a RREF.

i. Is v_2 in SPAN(v_1)? Yes/no. If yes, then write v_2 as a linear combination of v_1.

Answer: The RREF for this system is simply the first two columns in the RREF from part (a). That system is inconsistent because there is a pivot in the second column. So the answer is no.

ii. Is v_3 in SPAN(v_1, v_2)? Yes/no. If yes, then write v_3 as a linear combination of v_1, v_2.

Answer: RREF = first three columns of RREF in part (a). Consistent (no pivot in column 3) so the answer is yes. In the RREF we clearly see that Col3 = 2 Col1 + 1 Col2. That means $v_3 = 2v_1 + v_2$.

iii. Is v_4 in SPAN(v_1, v_2, v_3)? Yes/no. If yes, then write v_4 as a linear combination of v_1, v_2, v_3.

Answer: No (because there is a pivot in Column 4).

iv. Is v_5 in SPAN(v_1, v_2, v_3, v_4)? Yes/no. If yes, then write v_5 as a linear combination of v_1, v_2, v_3, v_4.

Answer: Yes. In the RREF we see Col5 = Col1 + 2 Col2 and so $v_5 = v_1 + 2v_2$.
(note: the answer v_1+2v_2 is not unique. Why? Because $v_1, \ldots, v_4$ are linearly dependent! Make sure you understand this, if not, check your notes or ask in class).

4. (a) (5 points). Give the definition: Vectors v_1, v_2, v_3 are linearly *dependent* when:

Must know: The definition in section 1.7 (first box on page 57).

Answer: when there exist x_1, x_2, x_3, not all 0, for which $x_1v_1 + x_2v_2 + x_3v_3 = 0$.

(b) (8 points). If v_1, v_2 are linearly dependent, must v_1, v_2, v_3 then also be linearly dependent? Yes/no. Explain using the definition from part (a).

v_1, v_2 dependent means there exist x_1, x_2, not all equal to 0, for which $x_1v_1 + x_2v_2 = 0$.

Then $x_1v_1 + x_2v_2 + 0v_3 = 0$ and not all of $x_1, x_2, 0$ are equal to 0. So v_1, v_2, v_3 are linearly dependent.

(c) (7 points). If A is a matrix and x is a vector, must Ax be a linear combination of the columns of A? Explain.

Must know: Formula for Ax in the box on page 35.

Yes. If $x_1, x_2, \ldots, x_n$ are the entries of the vector x, then $Ax = x_1\mathrm{Col}_1(A) + \cdots + x_n\mathrm{Col}_n(A)$ and this is a linear combination of the columns of A.