Linear algebra, test 2 answers

1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

   (a) (3 points). What is the rank of A? Answer: 1.
   (need to know: rank = number of pivots in RREF(A). Note: this A is already in RREF)

   (b) (5 points). Give a basis for the Column Space of A. Answer: $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$.
   (need to know: columns of A where RREF(A) has a pivot).

   (c) (12 points). Give a basis of the NullSpace of A.
   Answer: $\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\}$. (need to know: solve $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$, and that the free variables correspond to columns without a pivot, so they are x_2, x_3, and then you have to write down the solution in parametric form, and take the x_2 and x_3 components).

2. Let

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

   (a) (15 points). Compute the inverse of A.
   Answer: $\begin{pmatrix} 3 & -2 & -2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$
   (need to know: Row reduce $(A|I)$ to $(I|A^{-1})$)

   (b) (5 points) Use the inverse to solve the equation $AX = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.
   Answer: $\{\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ (need to know: Solution of $AX = b$ is $A^{-1}b$).

3. (12 points) Let V be a vector space and A is a matrix.

   (a) Give the definition: The NullSpace of A is the set of all...
   vectors v for which $Av = 0$.

   (b) Give the definition: $v_1, \ldots, v_n$ is a spanning set of V when:
   $\mathrm{SPAN}(v_1, \ldots, v_n) = V$
   (i.e.: the set of all linear combinations of $v_1, \ldots, v_n$ is V).

   (c) Give the definition: $v_1, \ldots, v_n$ is a basis of V when:
   It is a spanning set of V *and* it is linearly independent.

   (d) The dimension of V is defined as:
   the number of elements of a basis of V.

4. (7 points) Suppose that $v_1, v_2, v_3 \in V$ and that $B := \{v_1, v_2\}$ is a basis of V. Answer each with Yes/No/Undecidable and a brief explanation.

   (a) Is v_1, v_2, v_3 a spanning set of V?
   Yes. Since $v_1, v_2, v_3 \in V$ we have $\mathrm{SPAN}(v_1, v_2, v_3) \subseteq V$. But then $\mathrm{SPAN}(v_1, v_2, v_3)$ must equal V because it contains $\mathrm{SPAN}(v_1, v_2) = V$.

   (b) Is v_1, v_2, v_3 linearly independent?
   No. If v_1, v_2, v_3 were linearly independent, then with part (a) it would be a basis of V. But V can not have a basis with 2 elements and also have a basis with 3 elements.

5. (6 points) If A is an 5 by 7 matrix with rank 3, then what is:

   (a) The dimension of the column space of A? Answer: 3.
   (need to know: rank is always the dimension of the column space).

   (b) The dimension of the NullSpace of A? Answer: 4.
   (need to know: rank + dim(NullSpace) = number of columns)

6. Let

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and let $B = \{u_1, u_2, u_3\}$.

(a) (10 points) If $[v]_B = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ then what is v?

Answer: $1 \cdot u_1 + (-2) \cdot u_2 + 1 \cdot u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(b) (10 points) Let $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. What is $[e_2]_B$?

Row-reducing $(B|e_2) = (u_1 u_2 u_3 | e_2)$ leads to the answer $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$.

(c) (10 points): Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map and suppose that:

$$Tu_1 = u_2, \quad Tu_2 = u_3, \quad Tu_3 = u_1$$

Give the matrix of T.
Answer: we have to figure out what $T(e_1), T(e_2), T(e_3)$ are. Note that $e_1 = u_1 - 2u_2 + u_3$ (see part a) and $e_2 = u_2 - 2u_3$ (see part b) and $e_3 = u_3$. So $T(e_1) = T(u_1 - 2u_2 + u_3) = T(u_1) - 2T(u_2) + T(u_3) = u_2 - 2u_3 + u_1 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$, $T(e_2) = T(u_2) - 2T(u_3) = u_3 - 2u_1 = \begin{pmatrix} -2 \\ -4 \\ -5 \end{pmatrix}$, and $T(e_3) = T(u_3) = u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Thus the matrix is $(T(e_1)T(e_2)T(e_3)) = \begin{pmatrix} 1 & -2 & 1 \\ 3 & -4 & 2 \\ 3 & -5 & 3 \end{pmatrix}$.

7. (5 points): Answer each with True/False.
(Additional 5 bonus points): Explain.

(a) True or false: If A, B are square and $AB = I$ then A can be row-reduced to B?
Answer: True because if $AB = I$ then both are invertible, which means both can row-reduced to I, which means they can be row-reduced to each other.

(b) True or false: If A, B are square and A is invertible then AB and B have the same NullSpace?
Answer: True because if $BX = 0$ then $ABX = A0 = 0$, and if $ABX = 0$ then $A^{-1}ABX = 0$ and thus $BX = 0$.
Another way to see this is that if A is invertible, then it is a product of elementary matrices. So AB differs from B by a product of elementary matrices, which means AB can be row-reduced to B, and has thus the same solutions.

(c) True or false: If A is not square then A is automatically not invertible?
True. (Recall that if A has more rows than columns, then A can not be onto, and if it has more columns than rows, then it can not be one-to-one).

(d) True or false: If every column of B is in the NullSpace of A then $AB = 0$?
True: AB is a matrix whose j'th column is $A\mathrm{Col}_j(B)$ but that is 0 if $\mathrm{Col}_j(B)$ is in the NullSpace of A.

(e) True or false: $\{\begin{pmatrix} x \\ y \end{pmatrix} \mid x + y = 1\}$ is a vector space?
False because this does not contain the zero vector.