Linear algebra, test 2 (max score: 100 + 5 bonus)

1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

   (a) (3 points). What is the rank of A?

   (b) (5 points). Give a basis for the Column Space of A.

   (c) (12 points). Give a basis of the NullSpace of A.

2. Let

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

   (a) (15 points). Compute the inverse of A.

   (b) (5 points) Use the inverse to solve the equation $AX = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

3. (12 points) Let V be a vector space and A is a matrix.

   (a) Give the definition: The NullSpace of A is the set of all......

   (b) Give the definition: $v_1, \ldots, v_n$ is a spanning set of V when:

   (c) Give the definition: $v_1, \ldots, v_n$ is a basis of V when:

   (d) The dimension of V is defined as:

4. (7 points) Suppose that $v_1, v_2, v_3 \in V$ and that $B := \{v_1, v_2\}$ is a basis of V. Answer each with Yes/No/Undecidable and a brief explanation.

   (a) Is v_1, v_2, v_3 a spanning set of V?

   (b) Is v_1, v_2, v_3 linearly independent?

5. (6 points) If A is an 5 by 7 matrix with rank 3, then what is:

   (a) The dimension of the column space of A?

   (b) The dimension of the NullSpace of A?

6. Let

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and let $B = \{u_1, u_2, u_3\}$.

(a) (10 points) If $[v]_B = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ then what is v?

(b) (10 points) Let $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. What is $[e_2]_B$?

(c) (10 points): Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map and suppose that:

$$Tu_1 = u_2, \quad Tu_2 = u_3, \quad Tu_3 = u_1$$

Give the matrix of T.

7. (5 points): Answer each with True/False.
   (Additional 5 bonus points): Explain.

   (a) True or false: If A, B are square and $AB = I$ then A can be row-reduced to B?

   (b) True or false: If A, B are square and A is invertible then AB and B have the same NullSpace?

   (c) True or false: If A is not square then A is automatically not invertible?

   (d) True or false: If every column of B is in the NullSpace of A then $AB = 0$?

   (e) True or false: $\{\begin{pmatrix} x \\ y \end{pmatrix} \mid x + y = 1\}$ is a vector space?