Linear Algebra, Tuesday Nov 27, 2:00 - 3:15 pm, Answers

1. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$ and compute:

   (a) Answer: $\text{RREF}(A) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

   (b) Answer: rank(A)=1.

   (c) Explanation of answer: Computing a Basis for ColumnSpace(A) goes like this: take all columns of A (not of RREF(A)!) then throw away the ones that are linear combinations of prior columns. What's left is:

   $$\{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\}$$

   (d) Basis(NullSpace(A)): A few remarks

   - This is very similar to Test 2 question 1(c).
     See also the file `Test2_ANS.pdf`.
     Your lowest test-grade can be replaced by the grade on the final exam, but this won't help you if on the final you still can't compute the basis of a NullSpace.
   - NullSpace(A) means solving $Ax = 0$. The system $Ax = 0$ consists of these three equations:

     $0x_1 + 0x_2 + 0x_3 = 0$
     $1x_1 + 1x_2 + 1x_3 = 0$
     $2x_1 + 2x_2 + 2x_3 = 0$

     Our matrix A is simply the left-hand side of these three equations, without the + signs or the names x_1, x_2, x_3. Each row gives one equation, and the i'th column corresponds to the i'th variable x_i. Notice that part (a) simplified the above system of equations to:

     $1x_1 + 1x_2 + 1x_3 = 0$ (lets omit equations: $0x_1 + 0x_2 + 0x_3 = 0$)

     So whatever vectors you write down, please check that they are actually solutions of either the original system of equations $Ax = 0$, or the simplified system of equations $\text{RREF}(A)x = 0$.

- To solve the simplified system $x_1 + x_2 + x_3 = 0$ notice that we have two free variables x_2, x_3 and that the general solution is $\begin{pmatrix} -x_1 - x_2 \\ x_2 \\ x_3 \end{pmatrix}$. Taking the x_2 coefficient of that gives $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Do the same for the x_3 coefficient. The two resulting vectors are the basis for the NullSpace. So the answer is:

Answer:

$$\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \}.$$

It is *very* important that you can compute Basis(NullSpace). If not, you have to study that section again, because you'll need this in multiple places in the final exam.

The other thing you need to know is that NullSpace$(A - \lambda I)$ gives you the eigenvectors for eigenvalue λ. So NullSpace(A) gives you the eigenvectors for eigenvalue 0.

So at this point in the test, we found two independent eigenvectors for eigenvalue $\lambda = 0$.

(Note: an eigen*vector* must be $\neq 0$ but an eigen*value* can be 0).

For any eigenvalue λ, computing the eigenvectors for λ means computing a basis for NullSpace$(A - \lambda I)$.

(Note: if you don't find a non-zero element of NullSpace$(A - \lambda I)$, i.e., a non-zero solution of $(A - \lambda I)x = 0$, i.e. a non-zero solution of $Ax = \lambda x$, then λ is not an eigenvalue, in that case look for a computation error) (first check: sum(eigenvalues) = sum(diagonal), next check: see if you row-reduced $A - \lambda I$ correctly).

(e) The eigenvalues of A.

Write down $A - \lambda I = \begin{pmatrix} -\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 2 & 2 & 2-\lambda \end{pmatrix}$ and then compute its determinant. To compute that determinant you need to know one of the following methods, listing the easiest first:

i. Easiest: The formula on page 170 for the determinant of a 3 by 3 matrix.

ii. Medium: Theorem 1 on page 168 (use a cofactor expansion with respect to the first row).

iii. Hardest method: (in matrices without λ's this is usually the easiest method, but for matrices that contain λ this method is difficult and thus not recommended): Row-reduction to a triangular matrix (replacing Row 2 by Row 2 minus $1/(2-\lambda)$ times Row 3 makes the matrix lower triangular. Once the matrix is triangular, then the determinant is the product of the diagonal.

I recommend method knowing the formula on page 170 (i.e. method i). It's only for 3 by 3 matrices, but there won't be a determinant on the final that is bigger than 3 by 3.

If you use method ii, the co-factor expansion with respect to Row 1 is $-\lambda d_1 - 0d_2 + 0d_3$ where d_1 is the determinant of $\begin{pmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix}$.

Then we get $-\lambda d_1 = -\lambda\left((1-\lambda)(2-\lambda) - 1\cdot 2\right) = -\lambda(2-3\lambda+\lambda^2-2) = -\lambda(-3\lambda+\lambda^2) = -\lambda^2(\lambda-3)$.

The eigenvalues are the solutions of this polynomial, so we get $\lambda = 0, 0, 3$.

(f) Compute eigenvector(s) for each eigenvalue.

Computing the eigenvector(s) for λ means computing the NullSpace of $A - \lambda I$.

For $\lambda = 0$ we already did that in part (d).

For $\lambda = 3$ we get $A - \lambda I = \begin{pmatrix} -3 & 0 & 0 \\ 1 & -2 & 1 \\ 2 & 2 & -1 \end{pmatrix}$. If you row-reduce this and found no non-zero solution, then double-check your work. I row-reduced like this: replace R_1 by $-\frac{1}{3}R_1$. Then subtract R_1 from R_2. Then subtract $2R_1$ from R_3. Then add R_2 to R_3. Then divide R_2 by -2. Then we get $\text{RREF}(A-3I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$. Solution:

$\begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}$. Also OK is: $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ (it's OK to multiply an eigenvector by a non-zero scalar, because then it'll still be an eigenvector. But whatever you do please *do not guess* the solution of $\text{RREF}(A-3I)$. If you're not sure how to solve $\text{RREF}(A-3I)$ then please re-read part (d) above.

Next part of question (f):
Give a matrix P and a diagonal matrix D with $P^{-1}AP = D$.

P should contain the 3 independent eigenvectors we found (it doesn't matter in which order you place them, as long as you use the same ordering for the eigenvalues on the diagonal in matrix D). If I put them in the order I found them (i.e. the two vectors in (d), then the one we just found) then the corresponding eigenvalues are $0, 0, 3$, in which case $D = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 3 \end{pmatrix}$ and $P = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

2. Let A be a 4 by 3 matrix for which the rank is 2. (2 points each):

   (a) How many basic variables are there?
   Equal to the rank, so 2.

   (b) How many free variables?
   Equals number of variables minus the rank, so $3 - 2 = 1$.

   (c) Does the reduced row echelon form of A have rows that are all zero? If so, how many?
   The number should be the number of rows, minus the rank, so that is $4 - 2 = 2$.

   (d) True or false: The system $Ax = b$ is consistent for every $b \in \mathbb{R}^4$.
   False because there is a zero-row.

   (e) True or false: The system $Ax = 0$ has only the trivial solution (the zero solution).
   False because there is a free variable.

   (f) The linear map given by A is a map from $\mathbb{R}^3$ to $\mathbb{R}^4$

   (g) The linear map given by A is (choose one of the following):
   a) one-to-one but not onto.
   b) onto but not one-to-one.
   c) one-to-one and onto.
   d) neither one-to-one nor onto. THIS ONE. In part (d) we found "not onto" while part (e) we found "not one-to-one".

3. Let V be a vector space of dimension 3. True or false? (2 points each):

   (a) If 3 vectors in V are linearly independent then they must also be a spanning set.
   True (see the Basis Theorem in section 4.5).

   (b) A set of four vectors in V can never be a spanning set of V.
   False (if you don't see why, then read the Spanning Set Theorem in section 4.3).

   (c) If $T : V \to V$ is a one-to-one linear map then it must be onto as well.
   True: With coordinate-vectors (which we can use as soon as we choose a basis B of V) we have seen that a vector space V of dimension 3 is isomorphic to $\mathbb{R}^3$. Thus, T corresponds to a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$, and thus, a 3 by 3 matrix that is denoted as $[T]_B$.

The invertible matrix theorem says a lot of things about square matrices, among others, that one-to-one is equivalent to non-singular, and also equivalent to onto.

(d) An eigenvector is never zero.
True, that's part of the definition.

(e) If two matrices are similar, then they have the same (select one, or both, or neither): eigenvalues, eigenvectors.
We showed in class that they have the same eigenvalues (Theorem 4 in section 5.2). The eigenvectors are not the same (we showed that the matrix P in Theorem 4 sends the eigenvectors of one matrix to the eigenvectors of the other matrix).

4. Let $V = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 + x_2 + x_3 = 0\}$.
(V is the NullSpace of matrix A in Question 1 but this does not matter).
Let $T : V \to V$ be given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 \\ x_3 - x_2 \\ x_3 \end{pmatrix}$$

(a) Must any set $B = \{b_1, b_2\}$ for which $b_1, b_2 \in V$ are linearly independent must be a basis of V? Explain.
Yes. The Basis Theorem says that any set of p independent elements in a p-dimensional vector space form a basis.

(b) Let $B = \{b_1, b_2\}$ with $b_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

If $[u]_B = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ then what is u?

Answer: $-3b_1 + 1b_2 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$.

(c) If $v = \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}$ then what is $[v]_B$?

Observe that $v = 2b_1 - 7b_2$ which means $[v]_{\{b_1,b_2\}} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$.

(d) Suppose that $B' = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Compute the B to B' change of basis matrix is

$$B' \overset{P}{\leftarrow} B$$

and compute $[u]_{B'}$ (note: part (d) is unrelated to the later questions).

The formula for the B to B' change of basis is to row-reduce $(B'|B)$. Row-reducing that produced

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Then read off the answer from the right-upper corner $P = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. To test if we got the correct one, lets multiply $P[u]_B$ and check if we get $[u]_{B'}$. Note that $[u]_B$ is the vector in part (b). Multiplying P and $[u]_B$ I found $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. If the matrix P is correct, then that should equal $[u]_{B'}$. In other words u (computed in part (b)) should be 1 times the first element of B' plus 2 times the second element of B'. This checks out. So matrix P is correct.

(e) Compute $T(b_1)$ and $T(b_2)$.

$$T(b_1) = T\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 3 \cdot 0 \\ -1 - 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$T(b_2) = T\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 0 + 3 \cdot 1 \\ -1 - 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

Check if your result is in V!
OK: the formula $x_1 + x_2 + x_3 = 0$ (see definition of V in the first line of question 4) holds for these two vectors.

(f) Compute the coordinate vectors $[\;]_B$ of the vectors in part (e).
We have two write those last two vectors as $\ldots \cdot b_1 + \ldots \cdot b_2$ and then put the two $\ldots$ into a vector. We have to do that for $T(b_1)$ and for $T(b_2)$. Then we get $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(g) Compute matrix $[T]_B$.
Answer that means (1) applying T to B (which was question (e)), (2) compute their coordinate vectors w.r.t. B (that was question (f)), and then (3) put them into a matrix. Answer: $\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$

(h) Compute the eigenvalues of matrix $[T]_B$.
Lets write $A = [T]_B$ is our 2 by 2 matrix. The eigenvalues are the

roots of $\det(A - \lambda I)$. So compute $A - \lambda I$ and take the determinant, and we get $(2-\lambda)(-2-\lambda) - (-1)3 = -4 + \lambda^2 + 3 = \lambda^2 - 1$. The eigenvalues are its solutions: $1, -1$.

(i) For each eigenvalue compute an eigenvector of $[T]_B$.
You *must* be able to compute a NullSpace! If not, re-read exercise 1(d). Now take $A - \lambda I$, plug in $\lambda = 1$, and take the NullSpace (i.e. solve the equations). $A - 1I = \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}$. Now compute a basis of solutions (there is only 1 free variable, so we get 1 solution $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$. If you have any doubts, then you should double-check this: check that this vector really is a solution of $(A - 1I)x = 0$.
Next, $\lambda = -1$. Compute $A - (-1)I = \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$. Repeat the procedure, and I find $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(j) Use those eigenvectors to find eigenvectors of T.
You should remember that whenever you are using basis $B = \{b_1, b_2\}$, then no matter what b_1, b_2 are, whenever you have a vector like $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ then that represents $c_1 b_1 + c_2 b_2$. This means that our $\lambda = 1$ eigenvector represents $-3b_1 + 1b_2$ and our $\lambda = -1$ eigenvector represents $-1b_1 + 1b_2$. So the answer is:
$\lambda = 1$: $-3b_1 + 1b_2 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$. $\lambda = -1$: $-1b_1 + 1b_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

(k) Give a basis C of V such that $[T]_C$ is diagonal.
Answer is the same as part (j) because in part (j) we computed eigenvectors of T and if you use eigenvectors as your basis, then the matrix will be diagonal.