Linear Algebra, Test T3.pdf Question 4

To answer question 4, you need to understand pretty much all of the main goals in this course. So what are the main goals? They are:

(1) Understand any vector space, and

(2) Understand any linear map (short for: linear transformation).

1 Goal (1), understanding vector spaces

This V in question 4 is a plane lying inside $\mathbb{R}^3$. To be precise, the first line in question 4 says that V is the set of all vectors in $\mathbb{R}^3$ for which sum(entries)=0.

Here is the first thing you have to wrap your mind around:

(a) V is a plane, so in principle we should be able to denote any $v \in V$ with just 2 numbers.

(b) But V is a subset of $\mathbb{R}^3$, so any $v \in V$ is also an element of $\mathbb{R}^3$. So any $v \in V$ is a vector with 3 entries.

How do we square (a) with (b)? Well, every $v \in V$ has indeed 3 entries, but sum(entries) = 0 so if you only know 2 of those 3 entries then you can compute the 3'rd entry. For instance, the vector v in question 4(c), if I only gave you the first two entries 2 and -7 and then replaced the third entry with a question mark, it would not be a problem because you can still figure out that the third entry has to be 5 (because sum(entries)=0 for every $v \in V$).

So one way we can represent v is as follows: just keep the first two entries and throw the third one away. The reason this is OK is because as explained, if you're missing the third entry, you can always recompute it from the first two entries using the relation sum(entries) = 0 (recall that this relation was the very definition of V in the first line of question 4).

Now notice that the basis B was chosen in such a way that $[v]_B$ = first two entries of v! (check this! Compute $[v]_B$, and then try to understand why the result was a vector in $\mathbb{R}^2$ that simply contained the first two entries of v).

Also notice that this is kind of arbitrary, because instead of throwing away the last entry, we could equally well through away the first entry. Imagine that in your test, the first entry of v was not legible due to an ink stain. Well, since v is in V, we must have sum(entries)=0, and if you only know the last two entries (they are -7 and 5) then from sum(entries)=0 you can find out that the first entry must have been 2.

Notice that throwing away the first entry is precisely what B' does! To be precise $[v]_{B'}$ is simply the last two entries of v (check this too!).

But now we have multiple ways to represent the same vector $v \in V$. We could represent v with $[v]_B$ (which contains the first 2 entries of v). But we could also represent v with $[v]_{B'}$ (which contains the last 2 entries of v).

How are these vectors $[v]_B$ and $[v]_{B'}$ (both are in $\mathbb{R}^2$) related to each other? Well, $[v]_{B'}$ is some matrix times $[v]_B$. What is that matrix? Well, the matrix that sends $[\ldots]_B$ into $[\ldots]_{B'}$ is called the B-to-B' change of basis matrix, and you compute that matrix in question part (d). How do you compute this matrix? That's theorem 15 in the book but to work with that theorem, you also need to know how to compute coordinate-vectors because the matrix in formula (5) in theorem 15 is filled with coordinate-vectors.

Once you have that matrix, how do you test if it is correct? Well, multiply $[v]_B$ with your matrix and see if you get $[v]_{B'}$ (do this check).

Summary key ideas: Our V is a vector space of dimension 2. So it should be possible to represent any $v \in V$ with just 2 numbers. Such representation is not unique; it depends on a choice of a basis. Once you have chosen a basis, say B, then any $v \in V$ can be represented with $[v]_B$ and that will have 2 numbers in it because B has 2 elements.
In question 4, our basis B was chosen in such a way that $[v]_B$ is simply the first two entries of v. Why don't we always do that? Well, because that doesn't work for every vector space. Say for example $W = \{f | f'' = -f\}$, the set of all solutions to the differential equation $f'' = -f$. This is two-dimensional, so any $f \in W$ can be represented with 2 numbers. But which 2 numbers? That depends on the basis you choose. Say $f = 3\cos(x) + 2\sin(x)$. If you choose the basis $B_1 = \cos(x), \sin(x)$, then you can represent f as a vector with the numbers 3 and 2. But you could also choose the basis $B_2 = \sin(x), \cos(x)$. If you choose that basis, this same f would be represented by a vector with the numbers 2 and 3 (same numbers but in a different order because the basis has the same functions but in a different order). Which basis is the best one? There's not one answer to that, for one computation one basis is better, for other computations, another basis is better. (skip this part of you don't know complex numbers:) If we use complex numbers as our scalars, then we can also choose $B_3 := e^{ix}, e^{-ix}$ as our basis. Is that a better base or a worse basis? That depends on the application. Notice that differentiation sends e^{ix} to a scalar (namely: i) times e^{ix}. So e^{ix} is an eigenvector (with eigenvalue i) of the linear map "differentiation". Sometimes that is a big advantage, other times it doesn't matter.

The upshot is that in any vector space V, and for any basis B of V, we can represent any element $v \in V$ with a vector $[v]_B$ with this many entries: dimension(V).

Now back to question 4. You have to keep in mind that $[v]_B$ **represents** v but is not equal to v. Our $v \in V$ has 3 entries, but $[v]_B$ has only 2 entries (because $\dim(V)$ = number of elements of $B = 2$). So our vectors in V have 3 entries, but we *represent* them using vectors with 2 entries. So you should always keep in mind that if you have some $v \in V$, we can represent it with $[v]_B \in \mathbb{R}^2$. But conversely, if we have some element of $\mathbb{R}^2$, you have to keep

in mind that it *represents* an element of V and that that element of V has 3 entries.
Question 4(b) is about: given an element of $\mathbb{R}^2$, which element of V does it represent?
Question 4(c) is about: given an element v of V, what is the corresponding $[v]_B \in \mathbb{R}^2$ that we use to represent v with?
Question 4(d) is about: If instead of B we used B', then how are those elements of $\mathbb{R}^2$ related to one another?

2 Goal (2), understanding linear maps

This linear map T, it takes as input: an element of V (i.e. a vector with 3 entries and sum(entries) = 0). It gives as output: an element of V (also a vector with sum(entries) = 0).

So in part (e), how do you check that $T(b_1)$ is really in V? Well, just check if sum(entries) = 0.

We have now the following situation: T is given by an awkward formula using x_1, x_2, x_3. That's because our vectors $v \in V$ are in $\mathbb{R}^3$ so every $v \in V$ has 3 entries, and we just use x_1, x_2, x_3 to denote these entries (these entries satisfy the relation $x_1 + x_2 + x_3 = 0$ because we're only looking at elements of V).

What do I mean by "understanding linear maps"? Well, by that I mean that given this awkward description of T, you can figure out its true underlying structure, and that is the following:

- There is some line in V on which T acts in a very simple way (every element on that line gets sent to itself if you apply T) (it gets multiplied by 1) (it is an eigenvector of T with eigenvalue 1).
- Moreover, there is another line in V on which T also acts in a very simple (but slightly different) way, namely, applying T to elements of that line is the same as multiplying those elements by -1 (i.e. elements on that line are eigenvectors of T with eigenvalue -1).

Before I explain how we can figure out that "underlying structure" itemized here, let me reformulate the same claim in other words: I claim that V has some basis C_1, C_2 where C_1, C_2 are in V (that means they are vectors with 3 entries and with sum(entries)=0) such that T acts in a very simple way on C_1, C_2

$$T(C_1) = C_1 \quad \text{and} \quad T(C_2) = -C_2$$

So even though the original description of T was awkward, the map T is very simple if we write everything in terms of C_1, C_2
(What does it mean "write everything in terms of C_1, C_2"? Well, that means representing any vector $v \in V$ in terms of the coordinate-vector $[v]_{\{C_1, C_2\}}$).

For example $T(5C_1 + 8C_2)$ is easy to describe in terms of C_1, C_2 because all that T does is multiply C_1 by 1 (C_1 is an eigenvector with eigenvalue 1) and multiply C_2 by -1 (C_2 is an eigenvector with eigenvalue -1). So in this example, $T(5C_1 + 8C_2)$ is simply $5C_1 - 8C_2$. Since T is simple in terms of C_1, C_2 then that means that the matrix of T with respect to $\{C_1, C_2\}$ will be simple as well (it will be a diagonal matrix)

Now the main question is: Given the original awkward description of the linear map $T : V \to V$, how would I find out that there are $C_1, C_2 \in V$ in terms of which there is this much simpler description of T?

To do that, we first compute a 2 by 2 matrix for T. Then we compute two eigenvectors of that matrix. Those are in $\mathbb{R}^2$. Then keep in mind that any vector in $\mathbb{R}^2$ *represents* an element of V. So we can turn those two vectors in $\mathbb{R}^2$ to two vectors $C_1, C_2 \in V$.

How do we get a 2 by 2 matrix $[T]_B$ out of this awkward description for T? We did a number of examples of this in class last week, and also some the week before that, but many people were absent Monday/Tuesday before Thanksgivings, so here is the summary: To compute $[T]_B$ you have to:

Step (1) apply T to every element of B (this is exercise 4(e)). Then you get two elements of V. How do you check that they are really in V? Well, the first line in exercise 4 says: sum(entries) = 0.

Step (2): compute their coordinate vectors w.r.t. B. So you get $[T(b_1)]_B$ and $[T(b_2)]_B$. Remember that our B is very special, all that $[\ldots]_B$ really does is that it just throws away the last entry. In general though, you'd need to know how to compute coordinate-vectors. Now we get two vectors $[T(b_1)]_B$ and $[T(b_2)]_B$ in $\mathbb{R}^2$.

Step (3): put those vectors in a matrix (exercise 4(g)). Now you get a 2 by 2 matrix $[T]_B$ and that matrix tells you everything about T *in terms of* coordinate-vectors w.r.t. B. So whatever T does to vectors $v \in V$, the matrix $[T]_B$ does the same to their coordinate vectors. So to check if your matrix $[T]_B$ is right or not, you should take some $v \in V$, then compute $T(v)$, then take its coordinate vector $[T(v)]_B$. If your matrix $[T]_B$ is correct, then you could also get $[T(v)]_B$ by multiplying $[v]_B$ with your matrix $[T]_B$.

At this point, you now have a linear map T, with an awkward description in terms of entries x_1, x_2, x_3 of vectors in V. You also have a 2 by 2 matrix. Applying T to a vector v in V has the same effect as applying your matrix $[T]_B$ to the coordinate vector $[v]_B$ in $\mathbb{R}^2$. So basically the maps T and $[T]_B$ really do exactly the same thing except that they do that with different notation (notation $v \in V$ for T, and notation $[v]_B \in \mathbb{R}^2$ for $[T]_B$).

If you now compute the eigenvectors of $[T]_B$, you get two eigenvectors in $\mathbb{R}^2$.

These are not *equal* to elements of V, but they *represent* elements of V (In the same way as that $[u]_B \in \mathbb{R}^2$ in exercise 4(b) *represents* an element $u \in V$).

So doing the same as exercise 4(b) to your two eigenvectors of matrix $[T]_B$ in 4(i), then you get two eigenvectors in V (lets call them C_1, C_2) of the map T. And on that basis C_1, C_2 the map T will act in a very simple way, the matrix of T with respect to that basis is diagonal (with 1 and -1 on the diagonal) (if you have -1 and 1 on the diagonal that is also OK).

This question covers almost all the main material in the book so far, bases, vector spaces, linear maps, coordinate vectors, matrix of a linear map with respect to some basis, eigenvalues, eigenvectors, etc.
This is why, even though it is a lot to study, it is important to work on this question and turn it in as turn-in-homework, because by working through this question, you basically have to work through almost everything we've done so far, and that will be very helpful for the final exam.