Linear Algebra, Tuesday Nov 27, 2:00 - 3:15 pm
WRITE DOWN YOUR NAME:

1. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$ and compute:

   (a) (2 points). The reduced row echelon form of A.

   (b) (2 points). The rank of A.

   (c) (2 points). A basis for the column space of A.

   (d) (2 points). A basis for the null space of A.

   (e) (10 points). The eigenvalues of A (don't use the RREF from (a)!)

   (f) (10 points). Compute eigenvector(s) for each eigenvalue.
   Then give a matrix P and a diagonal matrix D with $P^{-1}AP = D$.

2. Let A be a 4 by 3 matrix for which the rank is 2. (2 points each):

   (a) How many basic variables are there?

   (b) How many free variables?

   (c) Does the reduced row echelon form of A have rows that are all zero? If so, how many?

   (d) True or false: The system $Ax = b$ is consistent for every $b \in \mathbb{R}^4$.

   (e) True or false: The system $Ax = 0$ has only the trivial solution (the zero solution).

   (f) The linear map given by A is a map from $\mathbb{R}^{\cdots}$ to $\mathbb{R}^{\cdots}$ (put numbers on the dots).

   (g) The linear map given by A is (choose one of the following):
   a) one-to-one but not onto.
   b) onto but not one-to-one.
   c) one-to-one and onto.
   d) neither one-to-one nor onto.

3. Let V be a vector space of dimension 3. True or false? (2 points each):

   (a) If 3 vectors in V are linearly independent then they must also be a spanning set.

   (b) A set of four vectors in V can never be a spanning set of V.

   (c) If $T : V \to V$ is a one-to-one linear map then it must be onto as well.

   (d) An eigenvector is never zero.

   (e) If two matrices are similar, then they have the same (select one, or both, or neither): eigenvalues, eigenvectors.

4. Let $V = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 + x_2 + x_3 = 0 \}$.
(V is the NullSpace of matrix A in Question 1 but this does not matter).
Let $T : V \to V$ be given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 \\ x_3 - x_2 \\ x_3 \end{pmatrix}$$

(a) (3 points). Must any set $B = \{b_1, b_2\}$ for which $b_1, b_2 \in V$ are linearly independent must be a basis of V? Explain.

(b) (3 points). Let $B = \{b_1, b_2\}$ with $b_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

If $[u]_B = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ then what is u?

(c) (3 points). If $v = \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}$ then what is $[v]_B$?

(d) (6 points). Suppose that $B' = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.
Compute the B to B' change of basis matrix is

$$B' \overset{P}{\leftarrow} B$$

and compute $[u]_{B'}$ (note: part (d) is unrelated to the later questions).

(e) (5 points) Compute $T(b_1)$ and $T(b_2)$. Check if your result is in V!

(f) (5 points) Compute the coordinate vectors $[\ ]_B$ of the vectors in part (e).

(g) (5 points). Compute matrix $[T]_B$. Hint: this should be a 2 by 2 matrix).

(h) (5 points). Compute the eigenvalues of matrix $[T]_B$. Hint: If you get square roots then first double-check your answer on the previous two questions and then on this question.

(i) (10 points). For each eigenvalue compute an eigenvector of $[T]_B$.

(j) (3 points). Use those eigenvectors to find eigenvectors of T.
Hint: see also part (b).

(k) (2 points). Give a basis C of V such that $[T]_C$ is diagonal.