Linear Algebra, Test 4 ANSWERS.

1. Let $B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. and let $C = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

   (a) (10 points). Compute the change of basis matrix from B to C.

   Write $B = b_1, b_2$ and $C = c_1, c_2$. The change-of-basis matrix from B to C is given by the formula in Theorem 15 on page 242:

$$([b_1]_C \ [b_2]_C)$$

   (the number n in Theorem 15 is the number of elements of B, also the number of elements of C, and that is $n = 2$ in this exercise).
   Our next task is to compute the coordinate vectors $[b_1]_C$ and $[b_2]_C$. Since the vectors are so small, we can find them with a bit of trial-and-error (try to write b_1 as a combination of c_1, c_2, then do the same for b_2) (writing b_1 as a combination of c_1, c_2 is particularly easy because $b_1 = 0c_1 + 1c_2$). Or, we can do it in a systematic way by row-reducing $(C|B) = (c_1 c_2 | b_1 b_2)$ to $\left(\begin{array}{c|c} I & P \\ 0 & 0 \end{array}\right)$.
   (That looks like the last formula on page 243, the reason they don't have any zero-rows at the bottom is because they have two vectors in $\mathbb{R}^2$ whereas we have 2 vectors in $\mathbb{R}^3$. In general, if B and C each have k vectors in $\mathbb{R}^N$ then you would get $N - k$ zero-rows at the bottom after row-reducing).

   Row-reducing $(C|B)$ gives $\left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$. From this we can read off $[b_1]_C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as we had already seen before, and $[b_2]_C = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ which is also easy to verify (check this!). So the B-to-C-change of matrix is $P = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}$.

   (b) (5 points). If $[w]_B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ then compute $[w]_C$ without computing w itself.

   P is the B-to-C change of basis matrix so it should send $[w]_B$ to $[w]_C$. That means $[w]_C = P[w]_B = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$

2. Let V be a vector space of dimension 3. True or false (2 points each):

(a) A set of four vectors in V can never be linearly independent. TRUE (if you have more than $\dim(V)$ vectors then they must be dependent).

(b) A set of four vectors in V can never be a spanning set of V. FALSE (a spanning set has at least $\dim(V)$ elements but it can easily have more by adding some unnecessary vectors).

(c) A set of two vectors in V can never be linearly independent. FALSE (the maximum number of independent vectors is equal to the dimension, but you can easily have fewer (just delete some!)).

(d) A set of two vectors in V can never be a spanning set of V. TRUE (a spanning set has at least $\dim(V)$ elements)

(e) Any three linearly independent vectors in V will always form a basis of V. TRUE (any independent set with $\dim(V)$ elements is a basis)

(f) A set of vectors in V can only be a spanning set of V if it contains three linearly independent vectors. TRUE (for every spanning set S there is a basis B with $B \subseteq S$. That basis will have $\dim(V)$ independent elements, but those will also be elements of S).

(g) A change of basis matrix is always invertible. TRUE.

3. Let $P_1 = \{a + bt \,|\, a, b \in \mathbb{R}\}$ be the vector space of all polynomials in t of degree at most 1. Let $T : P_1 \to P_1$ be the linear map given by differentiation $T = d/dt$.

(a) (10 points). Let $B = 1, t$ be a basis of P_1. Let $A = [T]_B$. Compute A.

To compute $[T]_B$ we have to apply T to all elements of B. After that, we have to compute the coordinate vectors w.r.t. B.
Writing $B = b_1, b_2 = 1, t$ we see that $T(b_1) = T(1) = 1' = 0$ and the coordinate vector of that w.r.t. B is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (our vector should have two entries because B has two elements). Now $T(b_2) = T(t) = t' = 1$ and the coordinate vector of that w.r.t. B is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Putting these two coordinate-vectors together we find

$$[T]_B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(b) (10 points). Compute all eigenvectors of A.

You may recognize this matrix as the first example in class today of a matrix that is not diagonalizable. Here is the short argument: The matrix is triangular, that means that the diagonal entries are the eigenvalues. So the eigenvalues are $0, 0$ (0 occurs twice). For the eigenvalue $\lambda = 0$ you compute the NullSpace of $A - \lambda I = A$

and the basis of that NullSpace is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since $\lambda = 0$ is our only eigenvalue, this means that all-together we've only found one linearly-independent eigenvector. But for an n by n matrix, diagonizable means having n independent eigenvectors. So we need 2, and found only 1. So this matrix is not diagonizable.

(c) (2 points). Is A diagonizable? No, see answer for (b).

(d) (2 points). Does there exist a basis C of P_1 for which $[T]_C$ is diagonal?

We haven't covered this in class yet, but if T is a linear map, and you compute its matrix $[T]_B$ w.r.t. some basis B, then compute the matrix $[T]_C$ w.r.t. another basis C, then $[T]_B$ and $[T]_C$ are similar. But according to part (c), the matrix $[T]_B$ is not diagonizable, which means, it is not similar to a diagonal matrix. But it is similar to $[T]_C$. That means that $[T]_C$ can not be a diagonal matrix.

4. V is a vector space with basis $B = b_1, b_2$ where $b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, b_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(a) (2 points). V is a ...-dimensional subspace of $\mathbb{R}^{\cdots}$ (put numbers on the dots).

V is a 2-dimensional subspace of $\mathbb{R}^3$

(b) (2 points). Let $T : V \to V$ be given by $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$.

Compute $T(b_1)$ and $T(b_2)$.

$T(b_1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, T(b_2) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

(c) (8 points). Compute the matrix $[T]_B$.

The vectors in part (b), we have to take their coordinate vectors w.r.t. basis B (this computation is similar to Exercise 1). We get

$$[T]_B = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}$$

(d) (10 points). Compute the eigenvectors of matrix $[T]_B$.

Since the matrix is triangular, we can simply read the eigenvalues from the diagonal: 1 and -1. Take the NullSpace of $[T]_B - 1I$ and we get the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $\lambda = 1$. Take the NullSpace of $[T]_B - (-1)I$ and we get the eigenvector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ for $\lambda = -1$.

(e) (5 points). Give a basis C of V for which $[T]_C$ is a diagonal matrix.

We have not yet covered this, but here is how it works. In part (d) we found the eigenvectors of $[T]_B$ and if we put those in matrix P then we have $P^{-1}[T]_B P = D$ for a diagonal matrix D. So here $P = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, the two columns are the eigenvectors we found, and the diagonal entries of D will be their eigenvalues (1 and -1).

We'll see in the next section that if P is the C-to-B change of basis matrix, then $P^{-1}[T]_B P = [T]_C$.

That means that if P is the C-to-B change of basis matrix, then $[T]_C$ is diagonal. But when is P the C-to-B change of basis matrix? Well, it would have to send $[w]_C$ to $[w]_B$

$$P[w]_C = [w]_B$$

If we take w to be the first element of C, then $[w]_C = e_1$ and then $[w]_B = Pe_1$ which is the first column of P. That column happens to be e_1 which tells us that w is the first element of B. So the first element of C is the first element of B.

Next, if we take w to be the second element of C, then $[w]_C = e_2$ and then $[w]_B = Pe_2$ which is the second column of P. Then w must be -2 times the first element of B plus 1 times the second element. Then $w = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Combined we see that $C = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

PS. Do you see that the matrix of the linear map T with respect to this basis C is indeed a diagonal matrix? In retrospect we should have expected this all along, because all that T does is switch two entries of our vectors (the first and third entries). That means that T is a reflection, which means it should have eigenvalues 1 and -1, and there should be some vector that stays put under this reflection, but there should also be a vector that gets multiplied by -1 under the reflection. But if swapping the first and third entries of a vector multiplies that vector by -1, then the first and third entries must have opposite signs, and the second entry must be zero. That's precisely what we found.

5. Let

$$A = \begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

(a) (6 points). Show that v_1 and v_2 are eigenvectors of A, and give the corresponding eigenvalues λ_1, λ_2.

$Av_1 = \begin{pmatrix} 9 \\ 3 \end{pmatrix} = 3v_1$ and $Av_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = -2v_2$, so they are eigenvectors with eigenvalues 3 and -2.

(b) (2 points). Compute the vectors $A^{14}v_1$ and $A^{14}v_2$ without computing any matrix-matrix or matrix-vector products, using only the fact that v_1, v_2 are eigenvectors and the fact that you know their eigenvalues from the previous question.

Matrix A simply multiplies v_1 by 3 so $A^{14}v_1 = 3^{14}v_1$.
Likewise, $A^{14}v_2 = (-2)^{14}v_2$.

(c) (6 points). Write the vector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a linear combination of v_1, v_2.

If we take $v_1 - v_2$ then we get a zero on the second entry (just like e_1 has) but we get a 5 on the first entry. So $e_1 = \frac{1}{5}v_1 - \frac{1}{5}v_2$.

(d) (6 points). Use the previous two questions to compute $A^{14}e_1$.

$A^{14}e_1 = \frac{1}{5}A^{14}(v_1 - v_2) = \frac{1}{5}(3^{14}v_1 - (-2)^{14}v_2)$.

(e) (2 bonus points, only do this exercise if you have time left). A petri dish contains bacteria that are either 0-day old or 1-day old. The situation is described by a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ where x is the number of 0-day old bacteria, and y is the number of 1-day old bacteria.
After every day, each 0-day old bacteria becomes 1-day old and produces one new 0-day old bacteria, this is described by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

while a 1-day old bacteria produces six new 0-day old bacteria and then dies, this is described by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 6 \\ 0 \end{pmatrix}.$$

Notice that this is precisely the action of matrix A. Suppose we start with one 0-day old bacteria and no 1-day old bacteria, then after 14 days, how many bacteria will there be?

ANSWER: Our starting vector is e_1, and after 14 days it will be $A^{14}e_1 = \frac{1}{5}(3^{14}v_1 - (-2)^{14}v_2) = \begin{pmatrix} 2876335 \\ 953317 \end{pmatrix}$ for a total of $2876335 + 953317$ bacteria.