Linear Algebra, Test 4, April 15, 2004.

1. Let $B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. and let $C = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

   (a) (10 points). Compute the change of basis matrix from B to C.

   (b) (5 points). If $[w]_B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ then compute $[w]_C$ without computing w itself.

2. Let V be a vector space of dimension 3. True or false (2 points each):

   (a) A set of four vectors in V can never be linearly independent.

   (b) A set of four vectors in V can never be a spanning set of V.

   (c) A set of two vectors in V can never be linearly independent.

   (d) A set of two vectors in V can never be a spanning set of V.

   (e) Any three linearly independent vectors in V will always form a basis of V.

   (f) A set of vectors in V can only be a spanning set of V if it contains three linearly independent vectors.

   (g) A change of basis matrix is always invertible.

3. Let $P_1 = \{a + bt \mid a, b \in \mathbb{R}\}$ be the vector space of all polynomials in t of degree at most 1. Let $T : P_1 \to P_1$ be the linear map given by differentiation $T = d/dt$.

   (a) (10 points). Let $B = 1, t$ be a basis of P_1. Let $A = [T]_B$. Compute A.
   (b) (10 points). Compute all eigenvectors of A.
   (c) (2 points). Is A diagonizable?
   (d) (2 points). Does there exist a basis C of P_1 for which $[T]_C$ is diagonal?

4. Let V be a vector space with basis $B = b_1, b_2$ where $b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, b_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(a) (2 points). V is a $\ldots$-dimensional subspace of $\mathbb{R}^{\cdots}$ (put numbers on the dots).

(b) (2 points). Consider the linear map $T : V \to V$ given by $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$.

Compute $T(b_1)$ and $T(b_2)$.

(c) (8 points). Compute the matrix $[T]_B$.

(d) (10 points). Compute the eigenvectors of matrix $[T]_B$.

(e) (5 points). Give a basis C of V for which $[T]_C$ is a diagonal matrix.

5. Let

$$A = \begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

(a) (6 points). Show that v_1 and v_2 are eigenvectors of A, and give the corresponding eigenvalues λ_1, λ_2.

(b) (2 points). Compute the vectors $A^{14}v_1$ and $A^{14}v_2$ without computing any matrix-matrix or matrix-vector products, using only the fact that v_1, v_2 are eigenvectors and the fact that you know their eigenvalues from the previous question.
(Hint: Multiplying A times v_1 is the same as multiplying λ_1 times v_1.
Hence, multiplying A times A times v_1 is the same as multiplying λ_1 times λ_1 times v_1.
So $A^2v_1 = AAv_1 = \lambda_1\lambda_1 v_1 = \lambda_1^2 v_1$. So then $A^{14}v_1$ is .....).

(c) (6 points). Write the vector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a linear combination of v_1, v_2.

(d) (6 points). Use the previous two questions to compute $A^{14}e_1$.

(e) (2 bonus points, only do this exercise if you have time left). A petri dish contains bacteria that are either 0-day old or 1-day old. The situation is described by a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ where x is the number of 0-day old bacteria, and y is the number of 1-day old bacteria.
After every day, each 0-day old bacteria becomes 1-day old and produces one new 0-day old bacteria, this is described by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

while a 1-day old bacteria produces six new 0-day old bacteria and then dies, this is described by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 6 \\ 0 \end{pmatrix}.$$

Notice that this is precisely the action of matrix A. Suppose we start with one 0-day old bacteria and no 1-day old bacteria, then after 14 days, how many bacteria will there be?