Linear algebra, test 3.

October 22, 2018

1. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \end{pmatrix}$$

(a) (5 points). Compute the reduced row echelon form of A.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) (8 points). Write down a basis B of $\text{Col}(A)$.

$$B = \{\text{Col}_1(A), \text{Col}_2(A)\} = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \}$$

(c) (12 points). Give a basis C of $\text{Nul}(A)$.

The general solution of $AX = 0$ can be read from the RREF, and we find:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

so we get this basis of the NullSpace

$$\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \}$$

(d) (10 points). Let v_1, v_2, v_3, v_4 be the columns of matrix A. What are: $[v_1]_B$, $[v_2]_B$, $[v_3]_B$ and $[v_4]_B$.

Do you see that the row-reduction needed to answer this question has already been done in part (a)? If so, then you can simply read off the answers:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

(e) (10 points). Let

$$w = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

What's the easiest way to tell if w is in $\text{Nul}(A)$ or not?

Answer: Just multiply Aw and see if it is zero.

If w is in $\text{Nul}(A)$, then compute its coordinate vector $[w]_C$ with respect to the basis C you computed in question (c).

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

I'll explain in class how to see this quickly.

2. Let $u = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. Let $V = \text{SPAN}\{u, v\}$ and $B = \{u, v\}$ a basis of V.

(a) (5 points). If w is some vector in V for which $[w]_B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ then what is w?

It is 1 times the first element of B plus -1 times the second element of B. So that equals $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$.

(b) (15 points). Which of the following vectors are in V? For each that is in V, give the coordinate vector with respect to B.

$$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

If we call these 5 vectors $w_1, \dots, w_5$, then take the augmented matrix $(uv|w_1 \dots w_5)$. Then row-reduce that and we get:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & -1 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

We then see that the augmented systems $(uv|w_2)$ and $(uv|w_5)$ are inconsistent, so w_2, w_5 are not in V. We also see the coordinate vectors w.r.t. B of w_1, w_3, w_4, namely, they are

$$[w_1]_B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, [w_3]_B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, [w_4]_B = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

Notice that w_1 is the same as w in the previous question, and we can see that this answer is consistent with the $[w]_B$ from the previous question.

(c) (3 points). Give a matrix A for which: $V = \text{Col}(A)$.

$$A = (uv) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{pmatrix}$$

(d) (12 points). Give a matrix N for which: $V = \text{Nul}(N)$.

Hint: To do this, you first need to compute a basis of the NullSpace of A^T (the transpose of A), and then you need to take the transpose of that. We'll finish that in class.

(e) (2 points). True or false: The vectors in question (b) form a spanning set for V?

FALSE: because some of the vectors in (b) are not in V.

(f) (2 points). True or false: If we take those vectors in question (b) that were in V, then we get a spanning set for V?

TRUE

(g) (2 points). True or false: If we take those vectors in question (b) that were in V, then we get a basis for V?

FALSE (we get too many vectors, i.e., we get linearly dependent vectors).

3. (10 points). For each of the following, mention if it is a vector space or not.
If it is not a vector space, then say why it is not a vector space by writing one of the following three things: "does not contain the zero vector", or "is not closed under addition", or "is not closed under scalar multiplication".
If it is a vector space, then no further explanation will be necessary.

   (a) $V = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$. VECTOR SPACE.

   (b) $V = \{\begin{pmatrix} a \\ b \end{pmatrix} \mid ab = 0\}$. NOT CLOSED UNDER ADDITION.

   (c) $V = \{\begin{pmatrix} a \\ b \end{pmatrix} \mid a + b + 3 = 0\}$. DOES NOT CONTAIN ZERO.

   (d) $V = \{\begin{pmatrix} a \\ b \end{pmatrix} \mid 2a - 5b = 0\}$. VECTOR SPACE

   (e) $V = \{\begin{pmatrix} a \\ b \end{pmatrix} \mid a \geq b\}$. NOT CLOSED UNDER SCALAR MULT. (to see this, take some elements of V and multiply them by -1).

4. (5 points). If $B = \{t^2 + t,\ t + 1,\ 1\}$ then what is $[t^2 - t + 1]_B$?

   To get $t^2 + \ldots$ we will need to take 1 of the first basis element of B. At this point we have $t^2 + t$ but to get $t^2 - t + \ldots$ we'll have to subtract 2 times the second basis element. Then we have $1 \cdot (t^2 + t) - 2 \cdot (t + 1) = t^2 - t - 2$ so to get the desired $t^2 - t + 1$ we have to add 3 times the third basis element: $t^2 - t + 1 = 1 \cdot (t^2 + t) - 2 \cdot (t + 1) + 3 \cdot 1$. Answer:

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$