Linear algebra, test 2, Feb 26 2004

1. (15 points). Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

   Give the matrix of T.

   You have to compute Te_1, Te_2, Te_3, those are the columns of the answer. So the first column of the answer is:

$$Te_1 = T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

   Compute Te_2 (i.e. $x_1 = 0, x_2 = 1, x_3 = 0$) and Te_3 (i.e. $x_1 = 0, x_2 = 0, x_3 = 1$) in a similar way and you get the second and third columns of the answer:

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

2. Let

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix}$$

   (a) (10 points). Compute the determinant of A.

   We will cover that in class today or tomorrow.

   (b) (10 points). Compute the inverse of A.

   Row-reduce $(A|I)$ to $(I|A^{-1})$. I found

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ -2 & 1 & -1 \end{pmatrix}$$

(c) (5 points). Use your answer of part (b) to solve:

$$AX = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Multiplying by A^{-1} tells us that

$$X = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-1 \\ -2+0 \\ -2-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix}.$$

3. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a 180^o rotation around the origin. So $S(v) = -v$ for every $v \in \mathbb{R}^2$.

(a) (5 points). Let A be the matrix of S. Compute A.

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) (5 points). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map "rotation around the origin with an angle of 90^o counter-clockwise". Let B be the matrix of T. Compute B.

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(c) (3 points). Explain without computing B^2 why $B^2 = A$.

Applying B means rotating 90^o so applying B twice (i.e. applying B^2) means rotating 180^o which is the same as applying A.

(d) (3 points). Explain without computing why $B^3 = B^{-1}$.

Applying B three times means rotating by 270^o which is the same as rotating -90^o which is the inverse of B (= rotating 90^o).

4. (20 points). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map and suppose that:

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Give the matrix of T. What is the inverse of this matrix?

Notice that if you apply T twice then it sends each vector back to itself! So T^2 is the identity, which means that $T = T^{-1}$. So whatever the matrix of T is, it is equal to its own inverse.

Now we have to compute the matrix of T. For that, we need to compute $T(e_1)$ and $T(e_2)$ and then put those two in a matrix. Now $T(e_1)$ is not given to us.
What is given are $T(u_1)$ and $T(u_2)$ where

$$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

In order to compute $T(e_1)$ from $T(u_1)$ and $T(u_2)$ we have to write e_1 as a linear combination of u_1, u_2. We find $e_1 = 2u_1 - u_2$ (how can you find those weights 2 and -1? Well, by rowreducing $(u_1 u_2 | e_1)$. Actually, it is best to rowreduce $(u_1 u_2 | e_1 e_2) = (u_1 u_2 | I)$ because then we'll also find the weights for the next one too). Now that we know that $e_1 = 2u_1 - u_2$ we can see that

$$T(e_1) = T(2u_1 - u_2) = 2T(u_1) - T(u_2) = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and we have found the first column of our matrix.

Similarly, $e_2 = -u_1 + u_2$ and so

$$T(e_2) = T(-u_1 + u_2) = -T(u_1) + T(u_2) = -\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and we found the second column of the matrix of T. So the matrix of T is

$$\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}$$

5. (24 points). True or false?

   (a) If $AB = AC$ and if $B \neq C$ then A can not be invertible.

   TRUE (because if you multiply $AB = AC$ (on the left!) by A^{-1} you get $B = C$. So if $B \neq C$ then that means we can't multiply by A^{-1}, which means that there is no A^{-1} (i.e. A is not invertible).

   (b) If A, B are square matrices and AB is the identity matrix then BA is also the identity matrix.

   TRUE (see page 114, theorem 8, as well as the last box on page 114)

   (c) If T is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^5$ then T is never one-to-one.

   FALSE

   (d) If T is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^5$ then T is never onto.

   TRUE

(e) If A is a square matrix, and $B = A^T$ is the transpose of A, and if B can not be row-reduced to the identity matrix then $AX = 0$ must have a non-trivial solution X.

TRUE: If A^T can not be row-reduced to I then it is not invertible, and then A is not invertible (use Theorem 8 multiple times).

(f) If A can be row-reduced to B then there exists an invertible matrix C such that $B = CA$.

TRUE (because row-reduction is the same as multiplying on the left by elementary matrices, and any product of elementary matrices is invertible).

(g) If A and B are square matrices, both not zero, then AB is also not zero.

FALSE

(h) If A is a square matrix and $\det(A) = 0$ then $AX = 0$ has only the trivial solution $X = 0$.

FALSE: The determinant is zero when A is singular (not invertible). But them Theorem 8 tells us that the equation $AX = 0$ must have a non-trivial solution.

(i) If A and B are square matrices, and if AB is invertible, then BA must also be invertible.

TRUE (if AB is invertible and A, B are square, then A and B must both be invertible)

(j) If A has more rows than columns, then the columns of A can not be linearly independent.

FALSE (if it has more columns than rows, then the columns can not be independent)

(k) If A is a m by n matrix and the reduced row echelon form has a zero row then $AX = 0$ has a non-trivial solution X.

FALSE (to get a non-trivial solution you need a column without a pivot) (having a row without a pivot doesn't tell us what we need to know here)

(l) If A can be row-reduced to the identity matrix then A^T (the transpose of A) can also be row-reduced to the identity matrix.

TRUE (if A can be row-reduced to I then A is invertible, but then so is A^T by Theorem 8).