Linear algebra, test 1, answers.

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1. Let

$$B = \begin{pmatrix} 2 & 1 & 1 & 4 & 7 \\ 1 & 1 & 2 & 5 & 8 \\ 0 & 1 & 3 & 6 & 9 \end{pmatrix}.$$

(a) Compute the rref (reduced row echelon form) of matrix B. Show which row operations you used. Use only *elementary* row operations.

Long answer: I started with $R_1 \leftrightarrow R_2$ to get the row-leader 1 at the top (this makes the computation much easier because if you have the row-leader 1 at the top then you don't have to divide by 2). Then I cleared out the rest of column 1 by doing $R_2 \leftarrow R_2 - 2R_1$. Then $R_3 \leftarrow R_3 + R_2$ and we reach row-echelon form.

We needed *reduced* row echelon form, so we're not yet done. We have to clean out column 2 (because it has a pivot, i.e. a row-leader). In addition, all row-leaders must be 1 (right now I have row-leader -1 in row 2 so row 2 has to be multiplied by -1). So I do $R_1 \leftarrow R_1 + R_2$ and $R_2 \leftarrow -R_2$ and I get the reduced row echelon form given below.

Short answer: $R_1 \leftrightarrow R_2$, $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_2$, $R_1 \leftarrow R_1 + R_2$, $R_2 \leftarrow -R_2$.

$$\text{rref}(B) = \begin{pmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 3 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note: You may have done the computation completely correctly and still have different row-operations than the ones listed above. However, if your row-operations are correct, then your row-operations must lead to the same rref(B) as my row-operations.

(b) If B is the augmented matrix of a system (3 equations, 4 unknowns and one right-hand side) then write down all solutions of this system.

Answer:
$x_1 = -1 + x_3 + x_4$
$x_2 = 9 - 3x_3 - 6x_4$
$x_3 = x_3$
$x_4 = x_4$ (so x_3, x_4 are free).

(c) Let v_i be the i'th column of B. So v_1 is column 1, v_2 is column 2, etc. Is v_3 in the SPAN of v_1, v_2?

Answer: Yes, because if take columns 1,2 in rref(B) as our coefficient matrix, and column 3 as our right-hand-side, we see that that is consistent, so $v_3 \in \text{SPAN}(v_1, v_2)$.

If so, then write v_3 as a linear combination of v_1, v_2.

Answer: We can immediately read this off from rref(B), but I'll explain how and why this works. Any linear relation that holds between the columns (an example of a relation that holds for the columns of matrix rref(B) is the following: "Col3 is -1 times Col1 plus 3 times Col2") is preserved under any of the three types of elementary row reductions. This relation "Col3 = –Col1 + 3 Col2" clearly holds in matrix rref(B) but that's a matrix that can be row-reduced to matrix B. So this same relation "Col3 = –Col1 + 3 Col2" is then also true for the original matrix B, but that just means $v_3 = -v_1 + 3v_2$, and there we have written v_3 as a linear combination of v_1, v_2.

(d) Does there exist a vector in $\mathbf{R}^3$ that is *not* in the SPAN of v_1, v_2, v_3, v_4, v_5?

Answer: Yes, because after row-reduction of $B = (v_1 \dots v_5)$ we got a zero-row. We can take a right-hand-side b such that after row-reduction of $(v_1 \dots v_5\ b)$ we get a zero-row in the coefficient side but a non-zero on the right-hand side. Then such b is not in the SPAN of v_1, v_2, v_3, v_4, v_5.

2. Consider the system of equations

$$1x_1 + 1x_2 + 1x_3 = 1$$

$$2x_1 + 1x_2 + 0x_3 = 2$$

$$2x_1 - 1x_2 + \alpha x_3 = 2$$

where x_1, x_2, x_3 are the unknowns, and α is some real number.

(a) Write down the augmented matrix of this system (note: the number α is considered as a coefficient, not as an unknown. So there are three equations in three unknowns, and we have one right-hand-side).

Answer:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & -1 & \alpha & 2 \end{pmatrix}.$$

(b) Row-reduce the system to upper triangular form (same as row echelon form. Note: you don't need to go as far as the reduced row echelon form).

Answer: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 2R_1$, $R_3 \leftarrow R_3 - 3R_2$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & \alpha+4 & 0 \end{pmatrix}.$$

Note: Your answer may look different. Remember that the *reduced* row-echelon-form is unique (so if we have to compute that, then everybody must get the same answer) but the row-echelon-form is not unique, your answer could be correct but look different.

(c) For which value(s) of α do we get free variable(s)? For those value(s) of α, write down all solutions $x_1 = \dots, \quad x_2 = \dots, \quad x_3 = \dots$.

Answer: Before answering questions like this or the next question, you should always keep your mind open for the following possibilities:

- None.
- One.
- Infinitely many.

For all of these kinds of questions, there's always either *none*, *one* or *infinitely many*. Never assume it's going to be just one, always keep your mind open for the other possibilities. OK, now lets look at the question: are there free variables?

Before we can answer that, we have to know when you get free variables. To pass this course, you'll need to know this: *A free variable corresponds to a column, without a row-leader, in the coefficient matrix of the (reduced) row-echelon-form.*

There are three columns in the coefficient matrix in the row-echelon-form above. Clearly, Col1 and Col2 have a row-leader (so x_1, x_2 are not free). Does Col3 have a row-leader? If $\alpha + 4 \neq 0$ then yes, and if $\alpha + 4 = 0$ then no. So, we have one free variable if $\alpha = -4$, and no free variables otherwise.

If $\alpha = -4$ then x_3 is free, from Row2 we get that $x_2 = -2x_3$. Now plug $x_2 = -2x_3$ into the first equation (= Row1), and we find $x_1 = 1 + x_3$.
NOTE: Don't write here $x_1 = 1 - x_2 - x_3$ because we don't want a basic variable like x_2 on the right-hand side. So if you wrote down $x_1 = 1 - x_2 - x_3$ then make use of the fact that we've already computed $x_2 = -2x_3$ in order to remove the basic variable x_2 from the right-hand side of your equation, and you get $x_1 = 1 - x_2 - x_3 = 1 - (-2x_3) - x_3 = 1 + x_3$.

(d) For which value(s) of α is the system consistent?

Answer: Again, keep an open mind. It could be: none, one, or infinitely many. Don't automatically assume something that it's one (even though for question (c) that would have worked OK). Why am I saying that. Well, because if you immediately assume there's going to be one, then you're going to try to calculate one value for α (but there may be none, or there may be infinitely many, I can always throw you a curve ball in these tests). To make sure you do it right, don't ask yourself "how am I going to compute α" but ask yourself this: "what exactly is being asked in this question"? Then you read the question, and you ask yourself: "when is a system consistent"? To pass this course, you'll need to know the answer to that: it's consistent if we do *not* get an inconsistent row in the row-echelon-form. Remember that an inconsistent row is a row that looks like this: $(0\ 0\ 0\ \ r)$ where the right-hand-side r is *not* zero.
Now look at the row-echelon-form and you see that it doesn't have an inconsistent row no matter what α is. Therefore, the system is consistent for every α.

For all those value(s) of α, give at least one solution $x_1 = \ldots, \quad x_2 = \ldots, \quad x_3 = \ldots$.

Answer: We've already given solutions when $\alpha = -4$. If $\alpha \neq -4$ then from Row3 we see that $x_3 = 0$. Then from Row2 we see $x_2 = 0$ and from Row1 we find that $x_1 = 1$. Clearly that's a solution for any α (whether α is -4 or not).

Clever answer: The system is this:
x_1 times Col1 plus x_2 times Col2 plus x_3 times Col3 equals rhs.
Here rhs is the right-hand-side. Now, if you had observed that rhs = Col1 then you would have immediately seen (without even looking at columns 2 and 3) that $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ is a solution. So we have a solution (and therefore: the system is consistent) no matter what α is, because we always have this solution $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ simply because rhs = Col1.

So remember this trick: If you see that rhs = Col1, then you immediately know a solution (so: consistent). But if you happen to notice that rhs equals 4 times Col1 minus 5 times Col3 (not in this exercise) then you know that $x_1 = 4$, $x_2 = 0$, $x_3 = -5$ is a solution.

3. Let A be the coefficient matrix of some system. The rank of matrix A is the number of pivots (the number of non-zero rows) in the (reduced) row echelon form of A. True or false:

(a) If the rank of A equals the number of *rows* of A then the system $Ax = b$ is consistent for every right-hand side b.
Answer: True, because if the number of rows is the number of pivots, then every row in the coefficient matrix has a pivot, so none of the rows in the coefficient matrix is zero, so we can't have any inconsistent rows.

(b) If the rank of A equals the number of *columns* of A then the system $Ax = b$ is consistent for every right-hand side b.
Answer: False. Just take A with more rows than columns, then we have more rows than pivots, so we will get zero row(s) in the coefficient matrix, and by taking a suitable right-hand-side we can make those inconsistent.

(c) The system $Ax = 0$ is always consistent (so here the right-hand side b is the zero vector).
Answer: True. To get an inconsistent row, we need zeros on the coefficient side with a non-zero on the right-hand-side. If the rhs is zero, we can't get that.

(d) If the rank of A is less than the number of *rows* of A then the system $Ax = 0$ always has free variables.
Answer: False. Take a matrix A with say: rank one, three rows, and one column. Then every column in A has a pivot, so we have no free variables. So we don't always get free variables.

(e) If the rank of A is less than the number of *columns* of A then the system $Ax = 0$ always has free variables.
Answer: True. If the number of pivots is the number of columns, then every column will have a pivot.

4. Find all (infinitely many) values of a and b for which vector w is in the SPAN of vectors v_1, v_2, v_3.

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad w = \begin{pmatrix} a \\ b \\ a \end{pmatrix}.$$

Clever Answer: Notice that in v_1, the sum of the entries is 0. And hey, that's true for v_2, and also for v_3. Well, then it must also be true for $v_1 + v_2$, for $5v_1$, for $-8v_3$ for $v_1 - 8v_3$, etc. It's going to be true for every linear combination of v_1, v_2, v_3. So if w is a linear combination of v_1, v_2, v_3, then we must have $a+b+a = 0$. Thus: $b = -2a$. Now I'll answer the question as if I hadn't noticed this.

Usual Answer: Row-reduce $(v_1\ v_2\ v_3\ \ w)$. The last row in the row-echelon-form is going to be this $(0\ 0\ 0\ \ 2a+b)$ (instead of $2a+b$ you could also have gotten $4a+2b$ or $-2a-b$ or something like that). If w is in the SPAN of v_1, v_2, v_3 we need the system to be consistent, so we need $2a+b$ to be zero, so $b = -2a$.
So the answer is: a is free and b is $-2a$.
Could also have said: b is free and a is $-b/2$.

Short answers:

1a. $\text{rref}(B) = \begin{pmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 3 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (note: there is only 1 correct answer)

$R_1 \leftrightarrow R_2$, $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_2$, $R_1 \leftarrow R_1 + R_2$, $R_2 \leftarrow -R_2$
(note: here there are many correct answers).

1b. $x_1 = -1 + x_3 + x_4, \quad x_2 = 9 - 3x_3 - 6x_4, \quad x_3 = x_3, \quad x_4 = x_4$

1c. Yes. $v_3 = -v_1 + 3v_2$.

1d. Yes.

2a and 2b. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & -1 & \alpha & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & \alpha+4 & 0 \end{pmatrix}$.
Note that since it's only row-echelon-form (not *reduced* row-echelon form) there are different correct answers for this second matrix.
Row-reductions: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 2R_1$, $R_3 \leftarrow R_3 - 3R_2$, but there are other correct answers.

2c. $\alpha = -4$. Then $x_1 = 1 + x_3, \quad x_2 = -2x_3, \quad x_3 = x_3$.

2d. Any α. $\quad x_1 = 1, \quad x_2 = 0, \quad x_3 = 0$.

3. True, False, True, False, True.

4. All a, b for which $2a + b = 0$.