Linear algebra, test 1

January 27, 2004

1. Let

$$B = \begin{pmatrix} 2 & 1 & 1 & 4 & 7 \\ 1 & 1 & 2 & 5 & 8 \\ 0 & 1 & 3 & 6 & 9 \end{pmatrix}.$$

   (a) Compute the rref (reduced row echelon form) of matrix B. Show which row operations you used. Use only *elementary* row operations.

   (b) If B is the augmented matrix of a system (3 equations, 4 unknowns and one right-hand side) then write down all solutions of this system.

   (c) Let v_i be the i'th column of B. So v_1 is column 1, v_2 is column 2, etc. Is v_3 in the SPAN of v_1, v_2? If so, then write v_3 as a linear combination of v_1, v_2.

   (d) Does there exist a vector in $\mathbf{R}^3$ that is *not* in the SPAN of v_1, v_2, v_3, v_4, v_5?

2. Consider the system of equations

$$1x_1 + 1x_2 + 1x_3 = 1$$

$$2x_1 + 1x_2 + 0x_3 = 2$$

$$2x_1 - 1x_2 + \alpha x_3 = 2$$

where x_1, x_2, x_3 are the unknowns, and α is some real number.

(a) Write down the augmented matrix of this system (note: the number α is considered as a coefficient, not as an unknown. So there are three equations in three unknowns, and we have one right-hand-side).

(b) Row-reduce the system to upper triangular form (same as row echelon form. Note: you don't need to go as far as the reduced row echelon form).

(c) For which value(s) of α do we get free variable(s)? For those value(s) of α, write down all solutions $x_1 = \ldots, \quad x_2 = \ldots, \quad x_3 = \ldots.$

(d) For which value(s) of α is the system consistent? For all those value(s) of α, give at least one solution $x_1 = \ldots, \quad x_2 = \ldots, \quad x_3 = \ldots.$

3. Let A be the coefficient matrix of some system. The rank of matrix A is the number of pivots (the number of non-zero rows) in the (reduced) row echelon form of A. True or false:

   (a) If the rank of A equals the number of *rows* of A then the system $Ax = b$ is consistent for every right-hand side b.

   (b) If the rank of A equals the number of *columns* of A then the system $Ax = b$ is consistent for every right-hand side b.

   (c) The system $Ax = 0$ is always consistent (so here the right-hand side b is the zero vector).

   (d) If the rank of A is greater than the number of *rows* of A then the system $Ax = 0$ always has free variables.

   (e) If the rank of A is greater than the number of *columns* of A then the system $Ax = 0$ always has free variables.

4. Find all (infinitely many) values of a and b for which vector w is in the SPAN of vectors v_1, v_2, v_3.

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad w = \begin{pmatrix} a \\ b \\ a \end{pmatrix}.$$