Linear algebra, review questions

1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

Compute each of the following products. If the product is not defined then mention that.

(a) AB

(b) BA

(c) AC

(d) CA

(e) BC

(f) CB

(g) BD

(h) DB

(i) CD

(j) DC

2. Let

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix}$$

   (a) Compute the inverse of A by Gauss-Jordan elimination.

   (b) Find a 3 by 2 matrix B such that

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 6 \end{pmatrix}$$

   Compute the following:

   (a) The reduced row echelon form of A

   (b) All solutions (the most general solution) of the following equations:

$$AX = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

   and

$$AY = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

   Verify at least one of your solutions.

4. Find all 2x2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which the following is true

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A = A \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

   Of the four variables a, b, c, d, how many are free variables?

5. True or false?

   (a) If A is row equivalent to B then the following two systems have the same solutions:

$$AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad BX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(b) If A is a square matrix and if A^2 is the zero matrix (all entries are zero) then A is also the zero matrix.

(c) If A is a square matrix then A is row equivalent to the identity matrix if and only if

$$AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has only one solution (the trivial solution).

(d) If A and B are invertible matrices of the same size then AB is an invertible matrix and the inverse of AB is $B^{-1}A^{-1}$.

6. The following is an invertible matrix:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$$

(a) Compute the inverse of A by row reduction.

(b) Solve $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ from the following equation:

$$A \cdot A \cdot X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(c) Use A^{-1} to solve $Y = \begin{pmatrix} y_1 & y_2 \end{pmatrix}$ from the following equation:

$$Y \cdot A = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

7. Let

$$B = \begin{pmatrix} -1 & -4 & 0 \\ 1 & 3 & 0 \\ -1 & -2 & 1 \end{pmatrix}.$$

(a) Compute $2B - B^2$.

(b) Use 7a to compute B^{-1}.

8.

$$C = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

(a) Compute the following: (Hint: rowreducing only one matrix is sufficient to answer all three questions 3b, 3c and 3d).

(b) The reduced row echelon form of C

(c) All solutions of the following equations:

$$CX = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

and

$$CY = \begin{pmatrix} 6 \\ 0 \\ 9 \end{pmatrix}.$$

Verify at least one of your solutions.

(d) Find all a and b for which

$$CZ = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

has a solution.

9. Consider the following system of linear equations

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + x_4 &= 3 \\ 2x_1 + 6x_2 + x_3 + x_4 &= 5 \\ 3x_1 + 10x_2 + x_3 + x_4 &= 9 \\ x_2 - 2x_3 - x_4 &= 1 \end{aligned}$$

(a) Give the augmented matrix for this system.

(b) Compute the reduced row echelon form.

(c) Compute all solutions (give the most general solution of this system).

10. The following is an invertible matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 1 & -1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$$

(a) Compute A^{-1}.

(b) Let X be the first column of A^{-1}.
Compute the product AX, and explain how you can use this as a check for the correctness of your A^{-1}.