Linear Algebra, Final, Monday April 26, 2004, 10:00 - noon

1. (4 points). Which of these vectors are eigenvectors of matrix $\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$?

   $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ yes ($\lambda = 1$), $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ no, $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ yes, ($\lambda = 0$), $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ no.

2. (9 points). Let A be a 3 by 4 matrix for which the rank is 3.

   - How many basic variables are there?

     That number always equals the rank. So the answer is 3.
   - How many free variables?

     That is the number of variables minus the rank, so that is $4 - 3 = 1$.
   - True or false: The reduced row echelon form of A has no zero-rows.

     True. The number of non-zero rows in RREF is the rank. That means all three rows are non-zero.
   - True or false: The system $Ax = b$ is consistent for every $b \in \mathbb{R}^3$.

     True because RREF has no non-zero rows.
   - True or false: The system $Ax = 0$ has only the trivial solution (the zero solution).

     False because there is a free variable.
   - The linear map given by A is a map from $\mathbb{R}^4$ to $\mathbb{R}^3$.
   - The linear map given by A is (choose one of the following):
     a) one-to-one but not onto.
     b) onto but not one-to-one. THIS ONE.
     c) one-to-one and onto.
     d) neither one-to-one nor onto.
   - If B is a n by m matrix and if the matrix product AB is defined, then which of the following must be true: (3 by 4) $\cdot$ (n by m) then:
     a) $n = 3$.
     b) $n = 4$. THIS ONE.
     c) $m = 3$.
     d) $m = 4$.
   - True or false: If V has dimension 3, then every set of 3 linearly independent elements of V will be a basis of V. TRUE.

3.

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh(x) = \sinh(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cosh(x)$$

Note: there is no minus sign in the derivative of $\cosh(x)$ because these are the *hyperbolic* trig functions and not the usual trig functions.
Let V be a vector space with basis $B = \{\cosh(x), \sinh(x)\}$.
Let $T : V \to V$ be the map given by differentiation, so $T = \frac{\mathrm{d}}{\mathrm{d}x}$.

(a) (8 points). Give the matrix $[T]_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) (4 points). Compute the eigenvectors of $[T]_B$.
$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda = 1$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \lambda = -1$.

(c) (2 points). Give the eigenvectors of T.
$\cosh(x) + \sinh(x)$, $\lambda = 1$ and $-\cosh(x) + \sinh(x)$, $\lambda = -1$.

(d) (1 point). Is $[T]_B$ diagonizable? YES.

(e) (1 point). Does there exist a basis C of V for which $[T]_C$ is diagonal.
YES: $b_1 = \cosh(x) + \sinh(x)$ and $b_2 = -\cosh(x) + \sinh(x)$.

4. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$ and compute:

(a) (2 points). The reduced row echelon form of A. RREF $= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) (2 points). The rank of A is: 1.

(c) (2 points). A basis for the column space of A is: $\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\}$.

(d) (4 points). A basis for the null space of A is: $\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}$.

(e) (8 points). The characteristic polynomial is $\lambda^2(\lambda - 1)$. For each eigenvalue, compute corresponding eigenvectors:
$\lambda = 0$: eigenvectors are given in part (d).
$\lambda = 1$: $A - \lambda I = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix}$. Basis(NullSpace) $= \{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\}$.
Give P such that $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Answer: $P = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

5. NOT YET COVERED:

6. (1 point). If u_1, u_2 and u are vectors, and $B = \{u_1, u_2\}$, then which matrix should you row-reduce in order to calculate $[u]_B$?
Answer: $(u_1 u_2 \mid u)$.

7. Suppose $B = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix}$ is the basis of some vector space V.

   (a) (2 points). The dimension of this V is 2.

   (b) (4 points). Let $u \in V$ and suppose $[u]_B = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. What is u?
   $u = 2\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ -15 \\ -11 \end{pmatrix}$.

   (c) (4 points). Suppose that C is another basis, and that the B to C change of basis matrix is $C \overset{P}{\leftarrow} B = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ What is $[u]_C$?
   Answer: We have $[u]_B$ in part (b) and the matrix that sends $[u]_B$ to $[u]_C$ so we just have to multiply $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ -9 \end{pmatrix}$.

   (d) (4 points). Compute the matrix $B \overset{P}{\leftarrow} C$.
   The C-to-B change-of-basis is the inverse of the B-to-C change-of-basis that was given in part (c). Inverting that we find $\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$.

   (e) (4 points). If $w \in V$ and $[w]_C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then what is $[w]_B$
   Answer: The matrix in part (d) sends $[w]_C$ to $[w]_B$. So $[w]_B = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$. Then $w = -5\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix}$.

   (f) (4 points). Let $v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Compute $[v]_B$.
   After row-reducing $(B \mid v)$ (first two columns = elements of B, third column = v) you find that the third column is 7 times the first column minus 3 times the second column. Then that must have also been true before you row-reduced, in other words $v = 7\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix}$ and thus $[v]_B = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$.

(g) (4 points). If $T\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ and $T\begin{pmatrix}2\\7\\5\end{pmatrix} = \begin{pmatrix}2\\7\\5\end{pmatrix}$ then what is $[T]_B$?

If we denote $B = b_1, b_2$ then you see from part (f) that $T(b_1) = 7b_1 - 3b_2$. Moreover, $T(b_2) = b_2$. Hence $[T]_B = \begin{pmatrix}7 & 0\\-3 & 1\end{pmatrix}$.

8. (7 points). Suppose T is a linear map for which we know the following:

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix}, \quad \text{and} \quad T\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}.$$

From this information, can you calculate $T\begin{pmatrix}1\\0\\-1\end{pmatrix}$?

YES, because $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ is a linear combination of $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}$
(if it wasn't, the answer to "can you calculate..." would have been NO).
Specifically, $\begin{pmatrix}1\\0\\-1\end{pmatrix} = 2\begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix}$
so $T\begin{pmatrix}1\\0\\-1\end{pmatrix} = 2T\begin{pmatrix}1\\1\\0\end{pmatrix} - T\begin{pmatrix}1\\2\\1\end{pmatrix} = 2\begin{pmatrix}2\\2\\0\end{pmatrix} - \begin{pmatrix}-1\\-2\\-1\end{pmatrix} = \begin{pmatrix}5\\6\\1\end{pmatrix}$.