Linear Algebra, Final, Monday April 26, 2004, 10:00 - noon
WRITE DOWN YOUR NAME:

1. (4 points). Which of these vectors are eigenvectors of matrix $\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$? For those that are, write the corresponding eigenvalue underneath that eigenvector.

$$\begin{pmatrix} 3 \\ 6 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. (9 points). Let A be a 3 by 4 matrix for which the rank is 3.
   - How many basic variables are there?
   - How many free variables?
   - True or false: The reduced row echelon form of A has no zero-rows.
   - True or false: The system $Ax = b$ is consistent for every $b \in \mathbb{R}^3$.
   - True or false: The system $Ax = 0$ has only the trivial solution (the zero solution).
   - The linear map given by A is a map from $\mathbb{R}^{\cdots}$ to $\mathbb{R}^{\cdots}$ (put numbers on the dots).
   - The linear map given by A is (choose one of the following):
     a) one-to-one but not onto.
     b) onto but not one-to-one.
     c) one-to-one and onto.
     d) neither one-to-one nor onto.
   - If B is a n by m matrix and if the matrix product AB is defined, then which of the following must be true:
     a) $n = 3$.
     b) $n = 4$.
     c) $m = 3$.
     d) $m = 4$.
   - True or false: If V has dimension 3, then every set of 3 linearly independent elements of V will be a basis of V.

3.

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh(x) = \sinh(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cosh(x)$$

Note: there is no minus sign in the derivative of $\cosh(x)$ because these are the *hyperbolic* trig functions and not the usual trig functions.
Let V be a vector space with basis $B = \{\cosh(x), \sinh(x)\}$.
Let $T : V \to V$ be the map given by differentiation, so $T = \frac{\mathrm{d}}{\mathrm{d}x}$.

(a) (8 points). Give the matrix $[T]_B$.

(b) (4 points). Compute the eigenvectors of $[T]_B$.

(c) (2 points). Give the eigenvectors of T.

(d) (1 point). Is $[T]_B$ diagonizable?

(e) (1 point). Does there exist a basis C of V for which $[T]_C$ is diagonal. If so, give such a basis.

4. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$ and compute:

(a) (2 points). The reduced row echelon form of A.

(b) (2 points). The rank of A is: ….

(c) (2 points). A basis for the column space of A.

(d) (4 points). A basis for the null space of A.

(e) (8 points). The characteristic polynomial is $\lambda^2(\lambda - 1)$. For each eigenvalue, compute corresponding eigenvectors, and give a matrix P such that $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

5. Let V be a vector space with the following basis:

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}.$$

(a) (10 points). Apply Gram Schmidt to obtain an orthogonal basis v_1, v_2, v_3 of V. If you have time left, then check that your v_1, v_2, v_3 are orthogonal!

(b) (3 points). Let $u = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Compute the orthogonal projection of u on V.

(c) (3 points). Let $u = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$. Compute the orthogonal projection of u on V.

(d) (3 points). Let $u = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}$. Compute the orthogonal projection of u on V.

(e) (3 bonus points). Find a basis of $V^\perp$.
Hint: You may use the fact that the dimension of $V^\perp$ is 1.

6. (1 point). If u_1, u_2 and u are vectors, and $B = \{u_1, u_2\}$, then which matrix should you row-reduce in order to calculate $[u]_B$?

7. Hint: If you don't know the answer to one question, you should still try the later questions because in most of the questions below you do not need the answer of the previous questions! Suppose $B = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix}$ is the basis of some vector space V.

   (a) (2 points). The dimension of this V is ....

   (b) (4 points). Let $u \in V$ and suppose $[u]_B = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. What is u?

   (c) (4 points). Suppose that C is another basis, and that the B to C change of basis matrix is $C \overset{P}{\leftarrow} B = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ What is $[u]_C$?

   (d) (4 points). Compute the matrix $B \overset{P}{\leftarrow} C$.

   (e) (4 points). If $w \in V$ and $[w]_C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then what is $[w]_B$ and what is w?

   (f) (4 points). Let $v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Compute $[v]_B$.

   (g) (4 points). If $T \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $T \begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix}$ then what is $[T]_B$?

8. (7 points). Suppose T is a linear map for which we know the following:

$$T\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\2\\0\end{pmatrix},\quad \text{and}\quad T\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}-1\\-2\\-1\end{pmatrix}.$$

From this information, can you calculate $T\begin{pmatrix}1\\0\\-1\end{pmatrix}$?