1. If V is a vector space, then $\{u_1, \ldots, u_n\}$ is a basis of V when all of the following are true:

   (a) $u_1, \ldots, u_n$ are in V.

   (b) Every element of V is a linear combination of $u_1, \ldots, u_n$

   (c) $u_1, \ldots, u_n$ is linearly independent.

2. If V is a vector space, and if we know its dimension $\dim(V)$ then it is less work to check if some set $\{u_1, \ldots, u_n\}$ is a basis of V or not, because in this case we can do the following:

   (a) First check that the number of vectors you have equals $\dim(V)$. If that's not the same number, it's not a basis.

   (b) Next, check that your vectors $u_1, \ldots, u_n$ are in V. In many exercises, that's already a given fact, but in some exercises you have to check that.

   (c) After that, you only need to check one out of these two (if one of them is true, then so is the other. And if one of them is false, then so is the other).

      i. Either check that every element of V is a linear combination of $u_1, \ldots, u_n$.

      ii. Or check that $u_1, \ldots, u_n$ is linearly independent.

   So you see that if you don't know $\dim(V)$ then you have to check both items 1b and 1c (same as items 2c.i and 2c.ii). But if you do know $\dim(V)$, which is often the case, then we have to check only one of these two items 1b, 1c.
   Furthermore, in many exercises it is easy to do one of these two checks but not so easy to do the other, so if you remember that if $\dim(V)$ is known that then one of the two checks (1b or 1c) is enough, then that can save quite a bit of time.
   In fact, in some exercises one of these two checks 1b or 1c might already be given as well as $\dim(V)$ so in those exercises you don't have to compute anything other than simply checking 2a and 2b which is usually easy to do.

3. Check for yourself that item 2 implies the following: Any n linearly independent elements of $\mathbf{R}^n$ form a basis of $\mathbf{R}^n$.

4. Some other useful things if you know $\dim(V)$.
    (a) If you have *more* than $\dim(V)$ vectors in V, then they're automatically linearly dependent.
    (b) If you have *fewer* than $\dim(V)$ vectors in V, then their SPAN can not be V (their SPAN will have to be smaller than V).
    (c) Note: in item 4a we can't say anything about the SPAN without doing computation.
    And in item 4b we can't say anything about whether they are dependent or independent without doing computation.
5. Dimensions. Let A be an m by n matrix, so m rows, and n columns.
    (a) $\dim(\,\mathrm{Col}(A)\,) = \mathrm{rank}(A)$.
    (b) $\dim(\,\mathrm{Row}(A)\,) = \mathrm{rank}(A)$.
    (c) $\dim(\,\mathrm{Nul}(A)\,) = n - \mathrm{rank}(A)$.
    (d) $\mathrm{Col}(A)$ is a subspace of $\mathbf{R}^m$ because each of the n columns of A has m entries, so each column of A is in $\mathbf{R}^m$.
    (e) The transpose of $\mathrm{Row}(A)$ is a subspace of $\mathbf{R}^n$ because each row has n entries.
    (f) $\mathrm{Nul}(A)$ is a subspace of $\mathbf{R}^n$ because if $v \in \mathrm{Nul}(A)$ then Av is zero, but to multiply A times v we need v to have n entries.
    (g) A^T, the transpose of A, is an n by m matrix.
    (h) $\mathrm{rank}(A^T) = \mathrm{rank}(A)$.
6. An n by n matrix A is invertible whenever one of these is true (then they're automatically all true).
    (a) The columns form a basis of $\mathbf{R}^n$.
    (b) The columns are linearly independent (see also item 3 from the previous page).
    (c) rank $= n$ (pivot in n columns)
    (d) $\mathrm{Col}(A) = \mathbf{R}^n$ (no zero-rows)
    (e) $\mathrm{Nul}(A) = \{0\}$ (no free variables, no columns without pivot).

    Note: If a matrix is not square, then d) and e) are not equivalent.

7. Some other useful facts:
   $\det(A^{-1}) = 1/\det(A)$
   $\det(A^T) = \det(A)$
   $\det(AB) = \det(A)\det(B)$
   Inverse of AB is $B^{-1}A^{-1}$.
   Transpose of AB is B^TA^T.
   $(A^{-1})^T = (A^T)^{-1}$ (transpose of A^{-1} = inverse of A^T).

8. If you have a polynomial equation $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$ then the *sum of the solutions equals* $-a_{n-1}$.
   And *the product of the solutions equals* $(-1)^n a_0$. Note that $(-1)^n a_0$ is just a_0 if n is even, and it is $-a_0$ if n is odd.

   For these two formulas to work, if you have a multiple solution, you must count it multiple times. An example for $n = 4$:
   $\lambda^4 - 11\lambda^3 + 42\lambda^2 - 68\lambda + 40$, this equals $(\lambda - 2)^3(\lambda - 5)$ so we have the solution $\lambda = 2$ (three times) and $\lambda = 5$ (one time).
   In the example $a_{n-1} = -11$ so $-a_{n-1} = 11$. This is indeed the sum of the solutions, if we count solution $\lambda = 2$ three times:
   $-a_{n-1} = 11 = 2 + 2 + 2 + 5$.
   Likewise, $(-1)^n a_0 = (-1)^4 40 = 40$ and this is indeed the product of the solutions, counting $\lambda = 2$ three times:
   $(-1)^n a_0 = 40 = (-1)^4 \cdot 2 \cdot 2 \cdot 2 \cdot 5$.

9. Let A be an n by n matrix. If you've computed the *characteristic equation* of A (the determinant of $A - \lambda I$) then check your characteristic equation as follows. Say you got: $\pm(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0) = 0$. (About that $\pm$, that's a $+$ if n is even and a $-$ if n is odd. Some texts use $\lambda I - A$ instead of $A - \lambda I$, in that case that $\pm$ is always just $+$).
   Now look at the number $-a_{n-1}$ (don't forget the $-$ sign). That number must be equal to the *sum of the diagonal of matrix* A. If it is not, you made a computation error.
   Then look at $(-1)^n a_0$. That must be the determinant of A. If it is not, then you've made a computation error.
   If your characteristic equation passes these two checks, then go ahead and solve it. The solutions are the eigenvalues. The sum of the solutions is $-a_{n-1}$ and the product is $(-1)^n a_0$, if we count multiple solutions multiple times as in the example above.

10. The previous handout LIST4.pdf described how to compute the eigenvectors of an n by n matrix A. The process described in item 6 of that handout always produces as many as possible *linearly independent* eigenvectors. Now look at item 3 of this handout, and we see that these eigenvectors form a basis of $\mathbf{R}^n$ if and only if we found n eigenvectors with this process. So: matrix A has a basis of eigenvectors if and only if the process in the previous handout produces n eigenvectors.

11. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbf{R}^n$ (so $e_1, \ldots, e_n$ are the columns of the identity matrix).

12. The simplest possible matrices are *diagonal matrices*. Why? Well, if D is a diagonal matrix then multiplying by D is very easy. Finding eigenvectors of D is also easy: If the entries on the diagonal are the numbers $\lambda_1, \ldots, \lambda_n$ then the vector e_i is an eigenvector of D with eigenvalue λ_i.

13. So a diagonal matrix has a basis of eigenvectors, namely $e_1, \ldots, e_n$. The corresponding eigenvalues can be read from the diagonal.

14. If M is a matrix and if each of these vectors $e_1, \ldots, e_n$ is an eigenvector of M, then M must be a diagonal matrix.

15. A matrix A is called diagonizable if there exists an *invertible* matrix P and a *diagonal* matrix D for which the following is true: $A = PDP^{-1}$.

16. A matrix A is diagonizable if and only if A has a basis of eigenvectors. So whether or not A is diagonizable is something we can figure out by doing the computations explained in the previous handout (LIST4.pdf). If we find n eigenvectors, then diagonizable, if we find fewer than n eigenvectors, then A is not diagonizable.

17. If $A = PDP^{-1}$ with D diagonal, and P invertible, then the *columns of P form a basis of eigenvectors of A.*

18. If A has a basis of eigenvectors, then put those eigenvectors as columns in the matrix P. Then put the corresponding eigenvalues (in the same ordering as you used in matrix P) on the diagonal of matrix D. Then PDP^{-1} will be equal to matrix A. If you've checked your eigenvectors (see item 1 in the previous handout LIST4.pdf, you should always check your eigenvectors in this way) then there is no need to check

that PDP^{-1} equals A, besides, multiplying PDP^{-1} would take up too much of your time anyway.

19. Many problems (see sections 5.6 and 5.7) are easy to solve for diagonal matrices. If a matrix A is not diagonal, then we can still solve those kind of problems by diagonalizing A (that means: computing the matrices P and D). Unfortunately, not every matrix is diagonizable. If a square matrix A is not diagonizable, then the next best thing is the so-called *Jordan normal form*. We do not have enough time in this course to compute the Jordan normal form, nevertheless, I do want you to know the following things:
    In the Jordan normal form, we'd write $A = PJP^{-1}$ where now J is not necessarily diagonal, but it is upper triangular, and moreover, it has as many as possible 0's. The Jordan normal form (matrix J) has 0's below the diagonal, the eigenvalues on the diagonal, the slanted line just above the diagonal has only 0's and 1's, and everything above that is 0.
    Problems like solving a system of linear differential equations can still be solved once we've computed the Jordan normal form. Remember just this then: The Jordan normal form is not necessarily a diagonal matrix, but it's as close to diagonal as one can possibly get. In applications of linear algebra you're likely to encounter the Jordan normal form, so it's a good thing if you've heard about it.

20. Upper triangular matrices. If A is an upper triangular matrix, then it's easy to compute the eigenvalues of A:
    The *eigenvalues* of an *upper triangular* matrix are simply the numbers on the diagonal.
    However, if you want to get the eigenvectors, you still need to do some work (see the previous handout LIST4.pdf).

21. If an n by n matrix has n distinct eigenvalues, then it is diagonizable. If it has fewer than n eigenvalues, then you'll have to calculate eigenvectors if you want to figure out if A is diagonizable or not.