Analytic Time-Domain Formulation for Acoustic Pressure Gradient Prediction in a Moving Medium

Chuan-Xing Bi,* Zhao-Huan Wang,† and Xiao-Zheng Zhang‡
Hefei University of Technology, 230009 Hefei, People's Republic of China

DOI: 10.2514/1.J055630

This paper presents an analytic time-domain formulation for acoustic pressure gradient prediction in a moving medium, which has significant application potential in evaluating the acoustic scattering boundary condition. Based on the convective Ffowcs Williams–Hawkings equation, a semianalytic time-domain acoustic pressure gradient formulation with a form involving the observer time differentiation outside the integrals is first developed, and then the desired analytic time-domain acoustic pressure gradient formulation is derived. Because the derived formulations are performed directly in the time domain, they are particularly applicable to the moving observer case. Simulation results for a stationary monopole source, a stationary dipole source, as well as a rotating monopole source in a moving medium demonstrate the effectiveness and accuracy of the proposed formulations for both stationary and moving sources with moving observers.

Nomenclature

A	=	amplitude of velocity potential, $\mathrm{m^2 \cdot s^{-1}}$
c_0	=	speed of sound in undisturbed medium, $\mathrm{m \cdot s^{-1}}$
f	=	data surface function
G	=	time-domain Green's function in a steady, uniform subsonic flow, $\mathrm{m^{-1}}$
H	=	Heaviside function
$\boldsymbol{L}$	=	source strength of the loading source, Pa
L_i	=	strength of the loading source components, Pa
L_M	=	$L_i M_i$, Pa
L_R	=	$L_i \tilde{R}_i$, Pa
L_{R^*}	=	$L_i \tilde{R}_i^*$, Pa
$\dot{L}_i$	=	$\partial L_i/\partial\tau$, $\mathrm{Pa \cdot s^{-1}}$
$\dot{L}_R$	=	$\dot{L}_i \tilde{R}_i$, $\mathrm{Pa \cdot s^{-1}}$
$\boldsymbol{M}$	=	source Mach number vector
M_i	=	components of source Mach number vector
M_R	=	$M_i \tilde{R}_i$
M_{R^*}	=	$M_i \tilde{R}_i^*$
$\boldsymbol{M}_\infty$	=	moving medium Mach number vector
$M_{\infty i}$	=	components of moving medium Mach number vector
$M_{\infty L}$	=	$M_{\infty i} L_i$, Pa
$M_{\infty M}$	=	$M_{\infty i} M_i$
$M_{\infty R}$	=	$M_{\infty i} \tilde{R}_i$
$M_{\infty R^*}$	=	$M_{\infty i} \tilde{R}_i^*$
$\dot{M}_i$	=	$\partial M_i/\partial\tau$
$\dot{M}_R$	=	$\dot{M}_i \tilde{R}_i$
n_i	=	components of unit vector normal to the data surface
p	=	pressure of local fluid, Pa
p_0	=	pressure of undisturbed medium, Pa
p'	=	sound pressure, Pa
Q	=	source strength of the thickness source, $\mathrm{kg \cdot m^{-2} \cdot s^{-1}}$
$\dot{Q}$	=	$\partial Q/\partial\tau$, $\mathrm{kg \cdot m^{-2} \cdot s^{-2}}$
R^*, R	=	acoustic radii, m
$\tilde{R}_i^*$	=	$\partial R^*/\partial x_i$
$\tilde{R}_i$	=	$\partial R/\partial x_i$
$\boldsymbol{r}$	=	geometrical vector between source and receiver; $\boldsymbol{x}$–$\boldsymbol{y}$, m
rs	=	rotating radius of source, m
S	=	data surface
T	=	source period, s
T_{ij}	=	Lighthill stress tensor, $\mathrm{kg \cdot m^{-1} \cdot s^{-2}}$
t	=	observer time, s
$\boldsymbol{U}_\infty$	=	velocity vector of moving medium, $\mathrm{m \cdot s^{-1}}$
$U_{\infty i}$	=	velocity vector components of moving medium, $\mathrm{m \cdot s^{-1}}$
$U_{\infty n}$	=	local normal velocity of moving medium, $\mathrm{m \cdot s^{-1}}$
$\boldsymbol{u}$	=	fluid velocity vector, $\mathrm{m \cdot s^{-1}}$
u_i	=	components of fluid velocity vector, $\mathrm{m \cdot s^{-1}}$
u_n	=	local normal velocity of fluid, $\mathrm{m \cdot s^{-1}}$
v_i	=	components of data surface velocity vector, $\mathrm{m \cdot s^{-1}}$
v_n	=	local normal velocity of data surface, $\mathrm{m \cdot s^{-1}}$
$\boldsymbol{x}$	=	observer position vector, m
(x_1, x_2, x_3)	=	Cartesian coordinate for the observer
$\boldsymbol{y}$	=	source position vector, m
(y_1, y_2, y_3)	=	Cartesian coordinate for the source
$\delta(\cdot)$	=	Dirac delta function
δ_{ij}	=	Kronecker delta
ρ	=	local fluid density, $\mathrm{kg \cdot m^{-3}}$
ρ_0	=	undisturbed medium density, $\mathrm{kg \cdot m^{-3}}$
ρ'	=	density perturbation of fluid, $\mathrm{kg \cdot m^{-3}}$
σ_{ij}	=	viscous stress tensor, $\mathrm{kg \cdot m^{-1} \cdot s^{-2}}$
τ	=	source time, s
$\varphi(\boldsymbol{x}, t)$	=	velocity potential function of source, $\mathrm{m^2 s^{-1}}$
ω	=	source pulsation angular frequency, $\mathrm{rad \cdot s^{-1}}$
ω_r	=	source rotating angular speed, $\mathrm{rad \cdot s^{-1}}$

Subscripts

e	=	calculation at retarded time
L	=	loading source
O	=	observer point
S	=	source point
T	=	thickness source

Received 25 August 2016; revision received 24 January 2017; accepted for publication 25 January 2017; published online 24 April 2017.

*Professor, Institute of Sound and Vibration Research, Anhui; cxbi@hfut.edu.cn.

†Ph.D. Candidate, Institute of Sound and Vibration Research, Anhui; zhaohuan@mail.hfut.edu.cn.

‡Associate Professor, Institute of Sound and Vibration Research, Anhui; xzhengzhang@hfut.edu.cn.

I. Introduction

THE acoustic scattering effect in many engineering applications, such as the scattering by a fuselage boundary layer [1–6], the rotor noise scattered by the centerbody [7–9], and the noise scattered by a centrifugal volute [10], should not be neglected because it substantially influences the overall noise in both magnitude and directivity [11].

Based on the solutions of the Ffowcs Williams–Hawkings (FW–H) equation [12] or the Kirchhoff formulation [13], numerical methods such as the boundary element method [4,14,15] and the equivalent source method [16–18] have been developed to predict the acoustic scattering field in recent years. When solving acoustic scattering problems, the key aspect is obtaining the acoustic velocity on the scattering surface to serve as the boundary condition. Recently, Ghorbaniasl et al. [19] suggested the analytic formulations V1 and V1A for calculating the acoustic velocity directly in the time domain, whereas the counterpart in the frequency domain was proposed by Mao et al. [20]. Given that the direct derivation of the acoustic velocity involves heavy algebraic manipulations, the acoustic pressure gradient can also be used as the boundary condition because it is related to the acoustic velocity through the acoustic velocity potential [21]. However, the direct numerical evaluation of the acoustic pressure gradient for a realistic scattering surface is computationally expensive; therefore, much research has been done to obtain the analytic pressure gradient formulation. Farassat and Brentner [22] derived a semianalytic formulation to calculate the acoustic pressure gradient. Lee et al. [23] first presented fully analytical formulation for the acoustic pressure gradient and implemented it into numerical codes. In that paper, the semianalytical formulation was revisited and named formulation G1, and the fully analytic formulation was named formulation G1A.

It should be noted that the medium was assumed stationary in the aforementioned studies. However, convection effects as in a wind-tunnel experiment may be important in aeroacoustic calculations. To realize the more complex acoustic scattering prediction for wind-tunnel experiments, the convective FW–H equation [24,25], which explicitly takes into account the presence of the moving medium, should be used and the acoustic scattering boundary condition in the moving medium should be calculated as well. Recently, Ghorbaniasl et al. [26] derived an analytic acoustic pressure gradient formulation in the frequency domain that accounted for the effect of a constant uniform flow with arbitrary direction. Considering that the time-domain formulation can be useful in some cases of acoustic scattering prediction (for example, the moving observer case), an analytic time-domain acoustic pressure gradient formulation that explicitly takes into account the presence of the moving medium is developed in the present paper. Inspired by the earlier work of Lee et al. [23], we will use similar names for our analytic formulation in a moving medium in the current paper: for example, G1A-M, in which M stands for a moving medium. This formulation can be seen as the extension of formulation G1A to a moving medium case. At the same time, semianalytic acoustic pressure gradient formulation G1-M is also given as part of the present study.

This paper is organized as follows. The convective FW–H equation and its time-domain solution are first briefly reviewed in Sec. II.A, and then the derivation of the formulation with a modified source term for acoustic pressure gradient is described in Sec. II.B. Subsequently, three numerical test cases are used to examine the performance of the proposed formulations in Sec. III. Finally, conclusions are drawn in Sec. IV.

II. Theory

A. Convective FW–H Equation and its Time-Domain Solution

Consider a uniform flow that moves at a constant velocity $\boldsymbol{U}_\infty$, and the direction of the velocity is arbitrary. Reorganizing the continuity and momentum equations that include the constant convective velocity term, the acoustic pressure at the observer $\boldsymbol{x}$ at time t could be described by the convective FW–H equation

$$\left[\frac{1}{c_0^2}\frac{\mathrm{D}^2}{\mathrm{D}t^2}-\nabla^2\right]\{p'(\boldsymbol{x},t)H(f)\}=\frac{\mathrm{D}}{\mathrm{D}t}[Q\delta(f)]-\frac{\partial}{\partial x_i}[L_i\delta(f)]+\frac{\partial^2}{\partial x_i\partial x_j}[T_{ij}H(f)] \tag{1}$$

with

$$\frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+U_{\infty i}\frac{\partial}{\partial x_i} \tag{2}$$

$$Q=\rho_0(v_n-U_{\infty n})+\rho[u_n-(v_n-U_{\infty n})] \tag{3}$$

$$L_i=[(p-p_0)\delta_{ij}-\sigma_{ij}]n_j+\rho u_i[u_n-(v_n-U_{\infty n})] \tag{4}$$

$$T_{ij}=\rho u_i u_j+[(p-p_0)-c_0^2(\rho-\rho_0)]\delta_{ij}-\sigma_{ij} \tag{5}$$

where $\delta(f)$ is the Dirac delta function; $H(f)$ is the Heaviside function; $f=0$ denotes the data surface; $U_{\infty n}=U_{\infty i}n_i$ with the local unit outer normal of $n_i=\partial f/\partial x_i$ in direction $x_i (i=1,2,3)$; δ_{ij} is the Kronecker delta; ρ_0, p_0, and c_0 are the density, pressure, and sound speed in the undisturbed medium, respectively; p is the local fluid pressure; ρ is the local fluid density; p' is the acoustic pressure; the local fluid velocity component is denoted by u_i; the local normal components to the data surface of the fluid and the body velocities are u_n and v_n, respectively; T_{ij} is the Lighthill stress tensor; and σ_{ij} is the viscous stress tensor.

The first two terms on the right-hand side of Eq. (1) are the monopole and dipole source terms, which are also known as the thickness and loading sources, respectively. The third term is the quadrupole source term, which is typically small compared to the other two terms when the fluid and moving body's velocities are both small; thus, it is reasonably omitted in the subsonic calculations. By neglecting the quadrupole source term, the integral solution of the convective FW–H equation was derived by Ghorbaniasl and Lacor in [27] as

$$p'(\boldsymbol{x},t,\boldsymbol{M}_\infty)=p'_T(\boldsymbol{x},t,\boldsymbol{M}_\infty)+p'_L(\boldsymbol{x},t,\boldsymbol{M}_\infty) \tag{6}$$

with the integral formulations over the data surface S:

$$\begin{aligned}4\pi p'_T(\boldsymbol{x},t,\boldsymbol{M}_\infty)=&\int_S\left[\frac{(1-M_{\infty R})\dot{Q}}{R^*(1-M_R)^2}\right]_e\mathrm{d}S-\int_S\left[\frac{c_0M_{\infty R^*}Q}{R^{*2}(1-M_R)}\right]_e\mathrm{d}S\\&+\int_S\left[(1-M_{\infty R})Q\frac{R^*\dot{M}_R+c_0(M_{R^*}-|\boldsymbol{M}|^2)}{R^{*2}(1-M_R)^3}\right]_e\mathrm{d}S\\&-\int_S\left[(1-M_{\infty R})Q\frac{c_0M_{R^*}M_R+c_0\gamma^2(M_{\infty M}^2-M_{R^*}^2)}{R^{*2}(1-M_R)^3}\right]_e\mathrm{d}S\\&-\int_S\left[\frac{c_0\gamma^2(M_{\infty R^*}M_{R^*}-M_{\infty M})Q}{R^{*2}(1-M_R)^2}\right]_e\mathrm{d}S\end{aligned} \tag{7}$$

and

$$\begin{aligned}4\pi p'_L(\boldsymbol{x},t,\boldsymbol{M}_\infty)=&\frac{1}{c_0}\int_S\left[\frac{\dot{L}_R}{R^*(1-M_R)^2}\right]_e\mathrm{d}S+\int_S\left[\frac{L_{R^*}-L_M}{R^{*2}(1-M_R)^2}\right]_e\mathrm{d}S\\&+\frac{1}{c_0}\int_S\left[L_R\frac{R^*\dot{M}_R+c_0\left(M_{R^*}-\left|\boldsymbol{M}\right|^2\right)}{R^{*2}(1-M_R)^3}\right]_e\mathrm{d}S\\&-\int_S\left[L_R\frac{M_{R^*}M_R+\gamma^2(M_{\infty M}^2-M_{R^*}^2)}{R^{*2}(1-M_R)^3}\right]_e\mathrm{d}S\\&-\int_S\left[\frac{L_{R^*}M_R+\gamma^2(M_{\infty M}M_{\infty L}-L_{R^*}M_{R^*})}{R^{*2}(1-M_R)^2}\right]_e\mathrm{d}S\end{aligned} \tag{8}$$

The mathematical background as well as the derivation procedure can be seen in [27]. In Eqs. (7) and (8), the acoustic radii R^* and R are

$$R^*=\frac{1}{\gamma}\sqrt{r^2+\gamma^2(\boldsymbol{M}_\infty\cdot\boldsymbol{r})^2} \tag{9}$$

$$R = \gamma^2(R^* - \boldsymbol{M}_\infty \cdot \boldsymbol{r}) \tag{10}$$

where

$$\gamma = \sqrt{1/(1 - \boldsymbol{M}_\infty^2)} \tag{11}$$

$$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y} \tag{12}$$

The quantities in brackets should be evaluated at the retarded time $\tau = t - R/c_0$, and the symbol $\boldsymbol{M}_\infty$ is the flow Mach number vector with $M_{\infty i} = U_{\infty i}/c_0$, where M is the magnitude of $\boldsymbol{M}$ that is the body Mach number vector with $M_i = v_i/c_0$. The other nomenclatures are defined as follows: $M_{\infty R} = M_{\infty i}\tilde{R}_i$, $M_{\infty R^*} = M_{\infty i}\tilde{R}_j^*$, $\tilde{R}_i^* = \partial R^*/\partial x_i$, $\tilde{R}_i = \partial R/\partial x_i$, $M_{\infty M} = M_{\infty i}M_i$, $M_R = M_i\tilde{R}_i$, $\dot{M}_R = \dot{M}_i\tilde{R}_i$, $M_{R^*} = M_i\tilde{R}_i^*$, $M_{\infty L} = M_{\infty i}L_i$, $L_M = L_iM_i$, $L_{R^*} = L_i\tilde{R}_i^*$, $L_R = L_i\tilde{R}_i$, and $\dot{L}_R = \dot{L}_i\tilde{R}_i$. The dots on several quantities denote derivatives with respect to the source time τ, and the dots on the main variables do not imply the differentiation of any of the associated vectors implied by the subscripts: for example, $\dot{L}_i = \partial L_i/\partial\tau$ and $\dot{M}_{R^*} = \dot{M}_i\tilde{R}_i^*$.

From a theoretical point of view, the analytic acoustic pressure gradient formulation can be derived by directly calculating the gradients of Eqs. (7) and (8); however, this process requires heavy mathematical operations. An alternative way to derive the analytic acoustic pressure gradient formulation will be presented in the next subsection, and this derivation is easily manipulated by adopting a modification of the source term at the beginning of the derivation.

B. Derivation Procedure of Formulations G1-M and G1A-M

Starting from the convective FW–H equation [Eq. (1)] and omitting the quadrupole source term, we can obtain a simplified convective FW–H equation as

$$\left[\frac{1}{c_0^2}\frac{\mathrm{D}^2}{\mathrm{D}t^2} - \nabla^2\right]\{p'(\boldsymbol{x},t)H(f)\} = \frac{\mathrm{D}}{\mathrm{D}t}[Q\delta(f)] - \frac{\partial}{\partial x_i}[L_i\delta(f)] \tag{13}$$

Employing Eq. (2) and adopting a modification of the source term recently suggested by Ghorbaniasl et al. [26],

$$F_i = L_i - QU_{\infty i} \tag{14}$$

Equation (13) can be further simplified to the following form:

$$\left[\frac{1}{c_0^2}\frac{\mathrm{D}^2}{\mathrm{D}t^2} - \nabla^2\right]\{p'(\boldsymbol{x},t)H(f)\} = \frac{\partial}{\partial t}[Q\delta(f)] - \frac{\partial}{\partial x_i}[F_i\delta(f)] \tag{15}$$

The Green's function used in a steady, uniform subsonic flow with the Mach number vector $\boldsymbol{M}_\infty$ is

$$G(\boldsymbol{x},t;\boldsymbol{y},\tau) = \frac{\delta(\tau - t + R/c_0)}{4\pi R^*} = \frac{\delta(g)}{4\pi R^*} \tag{16}$$

where

$$g = \tau - t + R/c_0 \tag{17}$$

Using the preceding Green's function yields the solution of the convective FW–H equation [Eq. (15)]:

$$p'(\boldsymbol{x},t,\boldsymbol{M}_\infty) = p'_\alpha(\boldsymbol{x},t,\boldsymbol{M}_\infty) + p'_\beta(\boldsymbol{x},t,\boldsymbol{M}_\infty) \tag{18}$$

where

$$4\pi p'_\alpha(\boldsymbol{x},t,\boldsymbol{M}_\infty) = \frac{\partial}{\partial t}\int_{-\infty}^{t}\int_S Q\frac{\delta(g)}{R^*}\,\mathrm{d}S\,\mathrm{d}\tau \tag{19}$$

$$4\pi p'_\beta(\boldsymbol{x},t,\boldsymbol{M}_\infty) = -\frac{\partial}{\partial x_i}\int_{-\infty}^{t}\int_S F_i\frac{\delta(g)}{R^*}\,\mathrm{d}S\,\mathrm{d}\tau \tag{20}$$

To obtain acoustic pressure gradient formulations, the gradient operation is performed to Eqs. (18–20), yielding

$$\frac{\partial p'(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} = \frac{\partial p'_\alpha(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} + \frac{\partial p'_\beta(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} \tag{21}$$

$$4\pi\frac{\partial p'_\alpha(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} = \frac{\partial}{\partial t}\frac{\partial}{\partial x_i}\int_{-\infty}^{t}\int_S Q\frac{\delta(g)}{R^*}\,\mathrm{d}S\,\mathrm{d}\tau \tag{22}$$

$$4\pi\frac{\partial p'_\beta(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} = -\frac{\partial^2}{\partial x_i^2}\int_{-\infty}^{t}\int_S F_i\frac{\delta(g)}{R^*}\,\mathrm{d}S\,\mathrm{d}\tau \tag{23}$$

Because the integral variables S and τ in Eqs. (22) and (23) are independent of the observer coordinates $x_i (i = 1, 2, 3)$, the gradient operators can be moved inside the integrals. Using the following equation that was derived in [27]

$$\frac{\partial}{\partial x_i}\left[\frac{\delta(g)}{R^*}\right] = -\frac{1}{c_0}\frac{\partial}{\partial t}\left[\frac{\tilde{R}_i\delta(g)}{R^*}\right] - \left[\frac{\tilde{R}_i^*\delta(g)}{R^{*2}}\right] \tag{24}$$

Equations (22) and (23) can be further rewritten as

$$4\pi\frac{\partial p'_\alpha(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} = -\frac{1}{c_0}\frac{\partial}{\partial t}\int_{-\infty}^{t}\int_S\frac{\partial}{\partial t}\left[\frac{Q\tilde{R}_i\delta(g)}{R^*}\right]\mathrm{d}S\,\mathrm{d}\tau - \frac{\partial}{\partial t}\int_{-\infty}^{t}\int_S\left[\frac{Q\tilde{R}_i^*\delta(g)}{R^{*2}}\right]\mathrm{d}S\,\mathrm{d}\tau \tag{25}$$

$$4\pi\frac{\partial p'_\beta(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} = \frac{1}{c_0}\frac{\partial}{\partial t}\int_{-\infty}^{t}\int_S F_j\frac{\partial}{\partial x_i}\left[\frac{\tilde{R}_j\delta(g)}{R^*}\right]\mathrm{d}S\,\mathrm{d}\tau + \int_{-\infty}^{t}\int_S F_j\frac{\partial}{\partial x_i}\left[\frac{\tilde{R}_j^*\delta(g)}{R^{*2}}\right]\mathrm{d}S\,\mathrm{d}\tau \tag{26}$$

To calculate the integral over $d\tau$, the identity of the generalized function [27,28] should be used:

$$\int_{-\infty}^{t} h(\tau)\delta(g)\,\mathrm{d}\tau = \left[\frac{h(\tau)}{|\partial g/\partial\tau|}\right]_{g=0} = \left[\frac{h(\tau)}{1 - M_R}\right]_e \tag{27}$$

With the help of Eq. (27), Eq. (25) can be written as

$$4\pi\frac{\partial p'_\alpha(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} = -\frac{\partial}{\partial t}E_1 \tag{28}$$

where

$$E_1 = \frac{1}{c_0}\frac{\partial}{\partial t}\int_S\left[\frac{Q\tilde{R}_i}{R^*(1 - M_R)}\right]_e\mathrm{d}S + \int_S\left[\frac{Q\tilde{R}_i^*}{R^{*2}(1 - M_R)}\right]_e\mathrm{d}S \tag{29}$$

To avoid the numerical evaluation of the observer time differentiation outside the integral in Eq. (29), they are converted to the source time differentiation through the following identity [27]:

$$\frac{\partial}{\partial t} = \left[\frac{1}{1 - M_R}\frac{\partial}{\partial\tau}\right]_e \tag{30}$$

E_1 then becomes

$$
\begin{aligned}
E_1 &= \frac{1}{c_0}\int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial\tau}\left[\frac{Q\tilde{R}_i}{R^*(1-M_R)}\right]_e \mathrm{d}S \\
&+ \int_S \left[\frac{Q\tilde{R}_i^*}{R^{*2}(1-M_R)}\right]_e \mathrm{d}S = \frac{1}{c_0}\int_S \left[\frac{\dot{Q}\tilde{R}_i}{R^*(1-M_R)^2}\right]_e \mathrm{d}S \\
&+ \int_S \left[Q\frac{-M_i+\gamma^2(M_{R^*}\tilde{R}_i^*-M_{\infty M}M_{\infty i})}{R^{*2}(1-M_R)^2}\right]_e \mathrm{d}S \\
&+ \frac{1}{c_0}\int_S \left[Q\tilde{R}_i\frac{R^*\dot{M}_R+c_0(M_{R^*}-M^2)}{R^{*2}(1-M_R)^3}\right]_e \mathrm{d}S \\
&- \frac{1}{c_0}\int_S \left[Q\tilde{R}_i\frac{c_0M_{R^*}(M_R-\gamma^2M_{R^*})+c_0\gamma^2M_{\infty M}^2}{R^{*2}(1-M_R)^3}\right]_e \mathrm{d}S \\
&+ \int_S \left[\frac{Q\tilde{R}_i^*}{R^{*2}(1-M_R)}\right]_e \mathrm{d}S
\end{aligned} \tag{31}
$$

Following the same steps used to obtain Eqs. (28) and (31), and employing an extra relation that is used to simplify the equations,

$$
\frac{\partial}{\partial x_i}\delta(g) = \frac{\tilde{R}_i}{c_0}\delta'(g) = -\frac{\tilde{R}_i}{c_0}\frac{\partial}{\partial t}[\delta(g)] \tag{32}
$$

Equation (26) would be further written as

$$
\begin{aligned}
4\pi\frac{\partial p_\beta'(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} &= \frac{1}{c_0}\frac{\partial}{\partial t}E_2 \\
&+ \int_S \frac{1/\gamma^2\cdot F_i + M_{\infty i}M_{\infty F} - 3\tilde{R}_i^*F_{R^*}}{R^{*3}(1-M_R)}\mathrm{d}S
\end{aligned} \tag{33}
$$

where

$$
\begin{aligned}
E_2 =& -\frac{1}{c_0}\int_S\left[\frac{\dot{F}_R\tilde{R}_i}{R^*(1-M_R)^2}\right]_e \mathrm{d}S \\
&- \int_S\left[\frac{\gamma^2\tilde{R}_i(M_{R^*}F_{R^*}-M_{\infty M}M_{\infty F})-F_M\tilde{R}_i}{R^{*2}(1-M_R)^2}\right]_e \mathrm{d}S \\
&- \int_S\left[\frac{\gamma^2F_R(M_{R^*}\tilde{R}_i^*-M_{\infty M}M_{\infty i})-F_RM_i}{R^{*2}(1-M_R)^2}\right]_e \mathrm{d}S \\
&- \frac{1}{c_0}\int_S\left[\frac{F_R\tilde{R}_iR^*\dot{M}_R+c_0F_R\tilde{R}_i(M_{R^*}-M^2)}{R^{*2}(1-M_R)^3}\right]_e \mathrm{d}S \\
&+ \int_S\left[\frac{F_R\tilde{R}_iM_{R^*}(M_R-\gamma^2M_{R^*})+\gamma^2F_R\tilde{R}_iM_{\infty M}^2}{R^{*2}(1-M_R)^3}\right]_e \mathrm{d}S \\
&+ \int_S\left[\frac{F_i+\gamma^2(M_{\infty i}M_{\infty F}-\tilde{R}_i^*F_{R^*})-\tilde{R}_i^*F_R-\tilde{R}_iF_{R^*}}{R^{*2}(1-M_R)}\right]_e \mathrm{d}S
\end{aligned} \tag{34}
$$

It should be noted that Eqs. (21), (28), and (33) are together called formulation G1-M, which can be seen as an extension of formulation G1 to the moving medium cases. Comparing formulation G1-M with the acoustic pressure formulation [Eqs. (6–8)], it is found that no more data are needed to calculate the acoustic pressure gradient than those used to predict the acoustic pressure in a moving medium; therefore, the two acoustic variables could be calculated at the same time. The observer time derivatives outside the integrals in Eqs. (28) and (33) can be evaluated numerically with various difference algorithms, such as the forward, backward, and central differences [29]. Compared with the direct numerical evaluation of the acoustic pressure gradient by using the acoustic pressure data of several observers, it is also an advantage that formulation G1-M does save considerable computing resources because the integral data of only one observer are needed.

The main drawback of formulation G1-M is that it is inconvenient to deal with the cases where the observer is not stationary. If the observer is stationary, the numerical observer time derivatives of the integrals in formulation G1-M are easy to deal with because the time history of the integrals in formulation G1-M can be obtained together with the acoustic pressure data at each observer time step. However, if the observer is moving, several extra evaluations of the integrals are needed to calculate the numerical observer time derivatives at each observer time step. To eliminate the numerical observer time derivatives of the acoustic pressure gradient calculation, an analytic formulation called G1A-M is deduced in the following.

The procedure for eliminating the observer time derivatives is to apply Eq. (30) to formulation G1-M and then evaluate the source time derivatives of the relevant variables. Inspired by the work of Lee et al. [23], some new functions and key source time derivatives are given in the following to make formulation G1A-M more concise:

$$
U(m,n) = \frac{1}{(R^*)^m(1-M_R)^n} \tag{35}
$$

$$
\begin{aligned}
V(m,n) &= \frac{\partial U(m,n)}{\partial\tau} \\
&= \frac{nR^*\dot{M}_R + nc_0\gamma^2(M_{R^*}^2-M_{\infty M}^2) + mc_0M_{R^*}(1-M_R) - nc_0M^2}{(R^*)^{m+1}(1-M_R)^{n+1}}
\end{aligned} \tag{36}
$$

$$
W = R^*\dot{M}_R + c_0(M_{R^*}-M^2) \tag{37}
$$

$$
\begin{aligned}
\dot{W} &= R^*\ddot{M}_R + c_0(\dot{M}_{R^*} - M_{R^*}\dot{M}_R - M_i\dot{M}_i) \\
&+ c_0\gamma^2(M_{R^*}\dot{M}_{R^*} - M_{\infty M}M_{\infty\dot{M}}) \\
&+ \frac{c_0^2\gamma^2(M_{R^*}^2-M_{\infty M}^2) - c_0^2M^2}{\gamma^2R^*}
\end{aligned} \tag{38}
$$

$$
Z = c_0M_{R^*}(M_R-\gamma^2M_{R^*}) + c_0\gamma^2M_{\infty M}^2 \tag{39}
$$

$$
\begin{aligned}
\dot{Z} &= \frac{(M_R-\gamma^2M_{R^*})[c_0R^*\dot{M}_R + c_0^2(\gamma^2M_{R^*}^2-M^2) - c_0^2\gamma^2M_{\infty M}^2]}{R^*} \\
&+ c_0M_{R^*}(\dot{M}_R-\gamma^2\dot{M}_{R^*}) + 2c_0\gamma^2M_{\infty M}M_{\infty\dot{M}}
\end{aligned} \tag{40}
$$

$$
B_i = -M_i + \gamma^2(M_{R^*}\tilde{R}_i^* - M_{\infty M}M_{\infty i}) \tag{41}
$$

$$
\begin{aligned}
\dot{B}_i =& -\dot{M}_i - \gamma^2M_{\infty\dot{M}}M_{\infty i} + \frac{\gamma^2R^*\dot{M}_{R^*}\tilde{R}_i^*}{R^*} \\
&+ \frac{c_0\gamma^2(2M_{R^*}^2\tilde{R}_i^* - M_{\infty M}^2\tilde{R}_i^* - M_{R^*}M_{\infty M}M_{\infty i}) - c_0(M^2\tilde{R}_i^* + M_{R^*}M_i)}{R^*}
\end{aligned} \tag{42}
$$

$$
A_i = \tilde{R}_i[\gamma^2(M_{R^*}F_{R^*} - M_{\infty M}M_{\infty F}) - F_M] \tag{43}
$$

$$
\begin{aligned}
\dot{A}_i &= \frac{\tilde{R}_i}{R^*}[\gamma^2R^*(F_{R^*}\dot{M}_{R^*} + M_{R^*}\dot{F}_{R^*})] \\
&+ \frac{c_0\gamma^2\tilde{R}_i(2M_{R^*}^2F_{R^*} - M_{\infty M}^2F_{R^*} - M_{\infty M}M_{\infty F}M_{R^*})}{R^*} \\
&+ \frac{\tilde{R}_i}{R^*}[-c_0(F_{R^*}M^2 + F_MM_{R^*}) \\
&- \gamma^2R^*(M_{\infty\dot{M}}M_{\infty F} + M_{\infty M}M_{\infty\dot{F}})] \\
&+ \frac{c_0B_i}{R^*}[\gamma^2(M_{R^*}F_{R^*} - M_{\infty M}M_{\infty F}) - F_M]
\end{aligned} \tag{44}
$$

$$D = R^* \dot{M}_R + c_0(M_{R^*} - M^2) \tag{45}$$

$$\dot{D} = -c_0 M_{R^*} \dot{M}_R + R^* \ddot{M}_R + c_0\gamma^2(M_{R^*}\dot{M}_{R^*} - M_{\infty M} M_{\infty \dot{M}}) - c_0 M_i \dot{M}_i + \frac{c_0}{\gamma^2 R^*}[\gamma^2 R^* \dot{M}_{R^*} - c_0 M^2 + c_0\gamma^2(M^2_{R^*} - M^2_{\infty M}) - 2\gamma^2 R^* M\dot{M}] \tag{46}$$

$$H = M_{R^*}(M_R - \gamma^2 M_{R^*}) + \gamma^2 M^2_{\infty M} \tag{47}$$

$$\dot{H} = \frac{(M_R - 2\gamma^2 M_{R^*})[\gamma^2 R^* \dot{M}_{R^*} - c_0 M^2 + c_0\gamma^2(M^2_{R^*} - M^2_{\infty M})]}{\gamma^2 R^*} + \frac{R^* M_{R^*}\dot{M}_R + c_0(\gamma^2 M^3_{R^*} - M_{R^*}M^2) - c_0\gamma^2 M_{R^*} M^2_{\infty M}}{R^*} + 2\gamma^2 M_{\infty M} M_{\infty \dot{M}} \tag{48}$$

$$K_i = F_i + \gamma^2(M_{\infty i} M_{\infty F} - \tilde{R}^*_i F_{R^*}) - \tilde{R}^*_i F_R - \tilde{R}_i F_{R^*} \tag{49}$$

$$\dot{K}_i = \dot{F}_i + \gamma^2 M_{\infty i} M_{\infty \dot{F}} - \dot{F}_{R^*}(\tilde{R}_i + \gamma^2 \tilde{R}^*_i) - \frac{[-c_0 F_M + c_0\gamma^2(M_{R^*}F_{R^*} - M_{\infty M} M_{\infty F})](\tilde{R}_i + 2\gamma^2\tilde{R}^*_i)}{\gamma^2 R^*} - \frac{c_0 B_i(F_R - 2\gamma^2 F_{R^*})}{\gamma^2 R^*} \tag{50}$$

It should be noted that the second partial derivative with respect to the source time is denoted by two dots over the quantity, and dots over the subscripts mean differentiation of the associated vectors implied by the subscripts: for example, $\ddot{M}_R = \ddot{M}\tilde{R}_i$ and $M_{\infty \dot{M}} = M_{\infty i}\dot{M}_i$. Taking the observer time derivatives inside the integrals of G1-M and using the aforementioned definitions, one obtains

$$4\pi \frac{\partial p'_\alpha(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = -\frac{1}{c_0}\int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[\dot{Q}\tilde{R}_i U(1,2)]_e \,\mathrm{d}S - \frac{1}{c_0}\int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[Q\tilde{R}_i U(2,3)W]_e \,\mathrm{d}S + \frac{1}{c_0}\int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[Q\tilde{R}_i U(2,3)Z]_e \,\mathrm{d}S - \int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[Q\tilde{R}^*_i U(2,1)]_e \,\mathrm{d}S - \int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[QB_i U(2,2)]_e \,\mathrm{d}S \tag{51}$$

and

$$4\pi \frac{\partial p'_\beta(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = -\frac{1}{c_0^2}\int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[\dot{F}_R\tilde{R}_i U(1,2)]_e \,\mathrm{d}S - \frac{1}{c_0}\int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[A_i U(2,2)]_e \,\mathrm{d}S - \frac{1}{c_0}\int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[F_R B_i U(2,2)]_e \,\mathrm{d}S - \frac{1}{c_0^2}\int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[F_R\tilde{R}_i D U(2,3)]_e \,\mathrm{d}S + \frac{1}{c_0}\int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[F_R\tilde{R}_i H U(2,3)]_e \,\mathrm{d}S + \frac{1}{c_0}\int_S \left[\frac{1}{(1-M_R)}\right]_e \frac{\partial}{\partial \tau}[K_i U(2,1)]_e \,\mathrm{d}S + \int_S (1/\gamma^2 \cdot F_i + M_{\infty i} M_{\infty F} - 3\tilde{R}^*_i F_{R^*}) U(3,1)\,\mathrm{d}S \tag{52}$$

The last step is to further rewrite Eqs. (51) and (52) as

$$4\pi \frac{\partial p'_\alpha(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = I_1 + I_2 + I_3 + I_4 + I_5 \tag{53}$$

$$4\pi \frac{\partial p'_\beta(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = I_6 + I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12} \tag{54}$$

where I_1 to I_{12} correspond to each of the integrals in Eqs. (51) and (52). The forms of I_i are given as follows:

$$I_1 = -\frac{1}{c_0}\int_S \left[\ddot{Q}\tilde{R}_i U(1,3) + \dot{Q}\dot{\tilde{R}}_i U(1,3) + \dot{Q}\tilde{R}_i V(1,2)U(0,1)\right]_e \mathrm{d}S \tag{55}$$

$$I_2 = -\frac{1}{c_0}\int_S \left[\dot{Q}\tilde{R}_i U(2,4)W + Q\dot{\tilde{R}}_i U(2,4)W + Q\tilde{R}_i V(2,3)U(0,1)W + Q\tilde{R}_i U(2,4)\dot{W}\right]_e \mathrm{d}S \tag{56}$$

$$I_3 = \frac{1}{c_0}\int_S \left[\dot{Q}\tilde{R}_i U(2,4)Z + Q\dot{\tilde{R}}_i U(2,4)Z + Q\tilde{R}_i V(2,3)U(0,1)Z + Q\tilde{R}_i U(2,4)\dot{Z}\right]_e \mathrm{d}S \tag{57}$$

$$I_4 = -\int_S \left[\dot{Q}\tilde{R}^*_i U(2,2) + Q\dot{\tilde{R}}^*_i U(2,2) + Q\tilde{R}^*_i V(2,1)U(0,1)\right]_e \mathrm{d}S \tag{58}$$

$$I_5 = -\int_S \left[\dot{Q}B_i U(2,3) + Q\dot{B}_i U(2,3) + QB_i V(2,2)U(0,1)\right]_e \mathrm{d}S \tag{59}$$

$$I_6 = -\frac{1}{c_0^2}\int_S \left[\left(\frac{\partial \dot{F}_R}{\partial \tau}\right)\tilde{R}_i U(1,3) + \dot{F}_R\dot{\tilde{R}}_i U(1,3) + \dot{F}_R\tilde{R}_i V(1,2)U(0,1)\right]_e \mathrm{d}S \tag{60}$$

$$I_7 = -\frac{1}{c_0}\int_S \left[\dot{A}_i U(2,3) + A_i V(2,2)U(0,1)\right]_e \mathrm{d}S \tag{61}$$

$$I_8 = -\frac{1}{c_0}\int_S \left[\left(\frac{\partial F_R}{\partial \tau}\right) B_i U(2,3) + F_R \dot{B}_i U(2,3) + F_R B_i V(2,2)U(0,1)\right]_e \mathrm{d}S \tag{62}$$

$$I_9 = -\frac{1}{c_0^2}\int_S \left[\left(\frac{\partial F_R}{\partial \tau}\right) \tilde{R}_i D U(2,4) + F_R \dot{\tilde{R}}_i D U(2,4)\right]_e \mathrm{d}S - \frac{1}{c_0^2}\int_S \left[F_R \tilde{R}_i \dot{D} U(2,4) + F_R \tilde{R}_i D V(2,3)U(0,1)\right]_e \mathrm{d}S \tag{63}$$

$$I_{10} = \frac{1}{c_0}\int_S \left[\left(\frac{\partial F_R}{\partial \tau}\right) \tilde{R}_i H U(2,4) + F_R \dot{\tilde{R}}_i H U(2,4) + F_R \tilde{R}_i \dot{H} U(2,4) + F_R \tilde{R}_i H V(2,3)U(0,1)\right]_e \mathrm{d}S \tag{64}$$

$$I_{11} = \frac{1}{c_0}\int_S \left[\dot{K}_i U(2,2) + K_i V(2,1)U(0,1)\right]_e \mathrm{d}S \tag{65}$$

$$I_{12} = \int_S (1/\gamma^2 \cdot F_i + M_{\infty i} M_{\infty F} - 3\tilde{R}_i^* F_{R^*}) U(3,1)\, \mathrm{d}S \tag{66}$$

It should be noted that $\dot{\tilde{R}}_i^*$, $\dot{\tilde{R}}_i$, and $\partial F_R/\partial \tau$ are defined as

$$\dot{\tilde{R}}_i^* = \frac{\partial \tilde{R}_i^*}{\partial \tau} = \frac{-c_0 M_i + c_0\gamma^2 (M_{R^*}\tilde{R}_i^* - M_{\infty M} M_{\infty i})}{\gamma^2 R^*} \tag{67}$$

$$\dot{\tilde{R}}_i = \frac{\partial \tilde{R}_i}{\partial \tau} = \gamma^2 \frac{\partial \tilde{R}_i^*}{\partial \tau} \tag{68}$$

$$\frac{\partial F_R}{\partial \tau} = \dot{F}_R + \frac{-c_0 F_M + c_0 \gamma^2 (M_{R^*} F_{R^*} - M_{\infty M} M_{\infty F})}{R^*} \tag{69}$$

Equations (21), (53), and (54), together with the definitions of I_i, are referred to as formulation G1A-M. Formulation G1A-M can be seen as an extension of formulation G1A to a moving medium case because it explicitly takes into account the effects of constant uniform flow. Compared with formulation G1-M, the observer time derivatives of the integrals in formulation G1A-M are no longer needed; thus, it is an advantage that only the time-dependent input data of the flowfield or (at most) numerical differentiation of them is required. Moreover, formulation G1A-M is more suitable to obtain the acoustic pressure gradient in cases where the observer is not stationary. However, it should be noted that a disadvantage of suggested formulation G1A-M is its mathematical complexity, in spite of the fact that some new functions and key source time derivatives are defined to make the expression of G1A-M concise.

III. Numerical Simulations

In this section, numerical simulations of three test cases in a moving medium are presented to validate the time-domain acoustic pressure gradient formulations developed in this paper. The first two test cases are the stationary monopole and dipole sources located in a moving medium with moving observers, whereas the third case consists of a rotating monopole with a moving observer for validating the corresponding moving source and moving observer case.

In the first two test cases, the stationary spherical surfaces are used as the data surfaces. The acoustic pressure gradient time history at the observer is evaluated and compared against the analytic solution. In the third test case, a moving spherical surface enclosing the monopole source is used as the data surface and the predicted acoustic pressure gradient time history at the observer is compared against the analytic solution. Moreover, the efficiency of formulations G1-M and G1A-M is compared in all three test cases.

To avoid any error related to flowfield simulation codes, all input flowfield data on the data surface are obtained from the analytic solutions of the flowfield generated by the sources. In this paper, the two-order central difference algorithm is performed to obtain the results from formulation G1-M.

A. Test Case 1: Monopole Source in a Moving Medium

The first test case is to consider a single-frequency monopole source located at the origin of a Cartesian coordinate system in a uniform flow with an arbitrary orientation.

The velocity potential for the monopole contains the uniform flow with an arbitrary direction defined as follows [27]:

$$\varphi(\boldsymbol{x}, t) = \frac{A}{4\pi R^*} \exp[i\omega(t - R/c_0)] \tag{70}$$

where the acoustic radii R^* and R have been defined in Sec. II.A. The acoustic particle velocity can be obtained from the gradient of the velocity potential

$$\boldsymbol{u}(\boldsymbol{x}, t) = \nabla\varphi(\boldsymbol{x}, t) \tag{71}$$

The induced acoustic pressure and density in a uniformly moving flow with an arbitrary direction are given by the unsteady Bernoulli equation:

$$p'(\boldsymbol{x}, t) = -\rho_0\left(\frac{\partial\varphi(\boldsymbol{x}, t)}{\partial t} + U_{\infty i}\frac{\partial\varphi(\boldsymbol{x}, t)}{\partial x_i}\right) \tag{72}$$

and

$$\rho'(\boldsymbol{x}, t) = p'(\boldsymbol{x}, t)/c_0^2 \tag{73}$$

The acoustic pressure gradient in the x_i direction is given by

$$\frac{\partial p'(\boldsymbol{x}, t)}{\partial x_i} = -\rho_0\left(i\omega\frac{\partial}{\partial x_i} + U_{\infty j}\frac{\partial^2}{\partial x_j \partial x_i}\right)\varphi(\boldsymbol{x}, t) \tag{74}$$

The analytic solutions of the acoustic pressure gradient at an observer point can be obtained through Eq. (74).

In this test case, the velocity potential amplitude of the monopole is $A = 1\ \mathrm{m^2/s}$. The angular frequency is $\omega = 10\pi\ \mathrm{rad/s}$. The ambient speed of sound c_0 is chosen as $340\ \mathrm{m/s}$. The freestream flow density ρ_0 is assumed to be $1.234\ \mathrm{kg/m^3}$. Two different mean flow Mach numbers of $\boldsymbol{M}_\infty = (0, 0, 0)$ and $\boldsymbol{M}_\infty = (0.6, 0.1, 0.5)$ are considered. The radius of the spherical data surface S is 1 m, and there are 15,292 triangular elements uniformly distributed on S for a fine enough spatial resolution. There are 30 time points used per source period ($T = 2\pi/\omega$) to ensure enough temporal resolution.

Figure 1 shows the predicted acoustic pressure gradient time history at the moving observer with different mean flow Mach numbers. The observer is moving along the x_1 axis at a constant

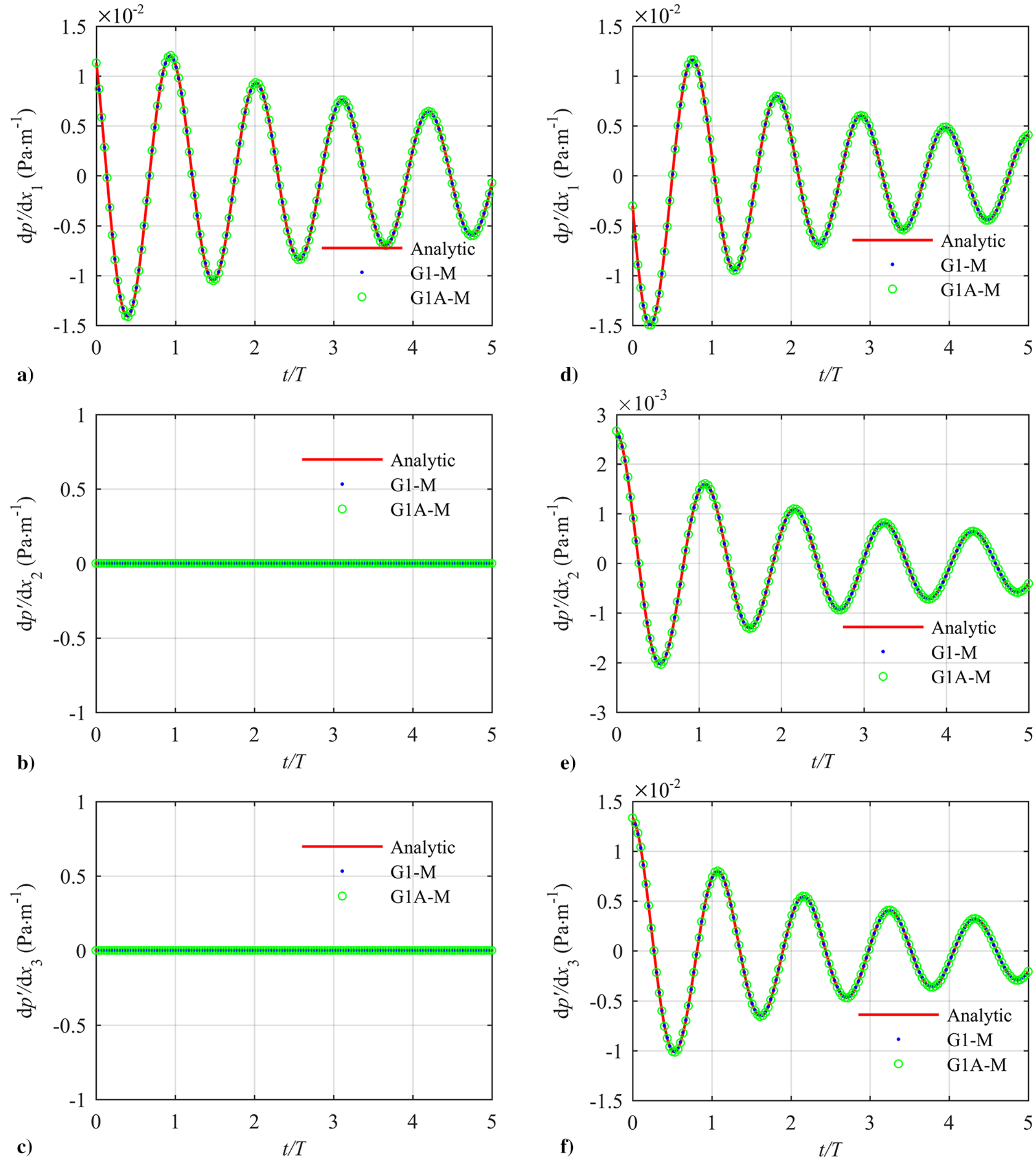

Fig. 1 Acoustic pressure gradient time history of a stationary monopole: a–c) $M_\infty = (0,0,0)$; d–f) $M_\infty = (0.6,0.1,0.5)$.

velocity of $v_o = 30$ m/s, and its initial position is $\boldsymbol{x} = (20, 0, 0)$m. It can be seen that the predicted results obtained from formulations G1A-M and G1-M both accurately match the analytic solutions, and thus the accuracy of the proposed time-domain formulations for the prediction of the acoustic pressure gradient is confirmed. When using a computer with an i7 CPU and 16 GB of memory, the computation time for formulation G1-M is 3.255 s and that for formulation G1A-M is 1.402 s. The reason why the computation time of formulation G1-M is longer than that of formulation G1A-M is that formulation G1-M needs to calculate several extra integrals at each time step when the observer is moving, and the differentiation calculation is time consuming.

B. Test Case 2: Dipole Source in a Moving Medium

The second test case is a dipole source located at the origin of a Cartesian coordinate system in a uniform flow with an arbitrary orientation, and the dipole axis is aligned with the x_2 axis. The velocity potential for such a dipole can be obtained by

$$\varphi(\boldsymbol{x}, t) = \frac{\partial}{\partial x_2}\left\{\frac{A}{4\pi R^*}\exp\left[i\omega\left(t - \frac{R}{c_0}\right)\right]\right\} \tag{75}$$

The procedure for obtaining the analytic acoustic pressure gradient is similar to that in the monopole source case. The following relations may be used in the calculation:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{R}_2^*}{\partial x_1}\right) = -\frac{(\partial \tilde{R}_2^*/\partial x_i)\cdot \tilde{R}_1^* + (\partial \tilde{R}_1^*/\partial x_i)\cdot \tilde{R}_2^* + (\partial \tilde{R}_2^*/\partial x_1)\cdot \tilde{R}_i^*}{R^*} \tag{76}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{R}_2^*}{\partial x_2}\right) = -\frac{(\partial \tilde{R}_2^*/\partial x_i)\cdot 2\tilde{R}_2^* + (\partial \tilde{R}_2^*/\partial x_2)\cdot \tilde{R}_i^*}{R^*} \tag{77}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{R}_2^*}{\partial x_3}\right) = -\frac{(\partial \tilde{R}_2^*/\partial x_i)\cdot \tilde{R}_3^* + (\partial \tilde{R}_3^*/\partial x_i)\cdot \tilde{R}_2^* + (\partial \tilde{R}_2^*/\partial x_3)\cdot \tilde{R}_i^*}{R^*} \tag{78}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{R}_2}{\partial x_1}\right) = \gamma^2 \frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{R}_2^*}{\partial x_1}\right) \tag{79}$$

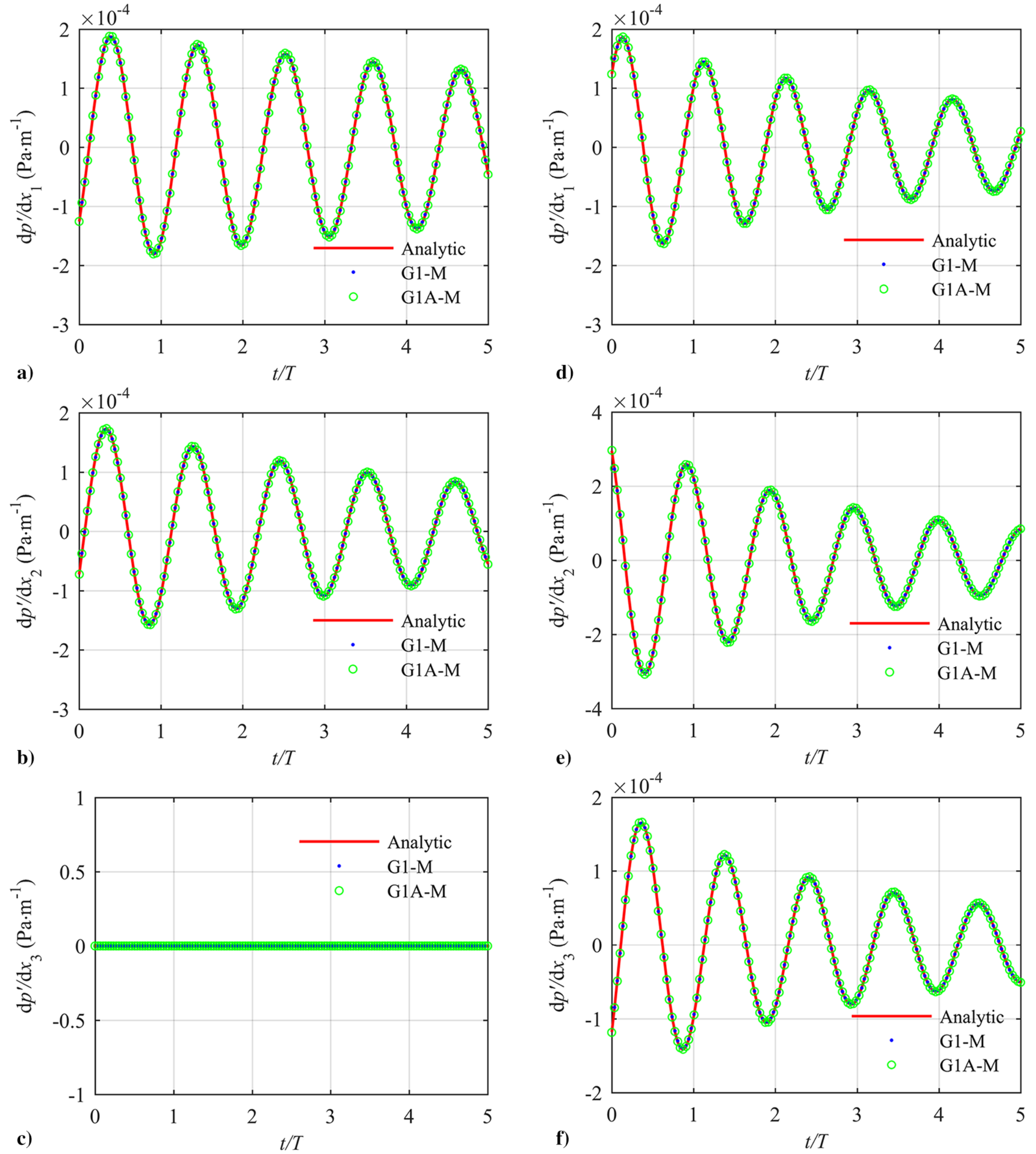

Fig. 2 Acoustic pressure gradient time history of a stationary dipole: a–c) $M_\infty = (0,0,0)$; d–f) $M_\infty = (0.8,0.1,0.4)$.

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{R}_2}{\partial x_2}\right) = \gamma^2 \frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{R}_2^*}{\partial x_2}\right) \tag{80}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{R}_2}{\partial x_3}\right) = \gamma^2 \frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{R}_2^*}{\partial x_3}\right) \tag{81}$$

where the spatial derivatives on the right-hand sides of Eqs. (76–78) have been defined in Sec. II.A.

In this test case, two different mean flow Mach numbers of $\boldsymbol{M}_\infty = (0, 0, 0)$ and $\boldsymbol{M}_\infty = (0.8, 0.1, 0.4)$ are considered. The spherical data surface together with its mesh data used here are the same as those in the first test case; parameters A, ω, c_0, and ρ_0, as well as the time sampling points in one source period, are also set to the same values as those in the first test case.

Figure 2 shows the predicted acoustic pressure gradient time history at the moving observer, which is moving along the x_1 axis at a constant velocity of $v_o = 30$ m/s, and its initial position is $\boldsymbol{x} = (50, 50, 0)$m.

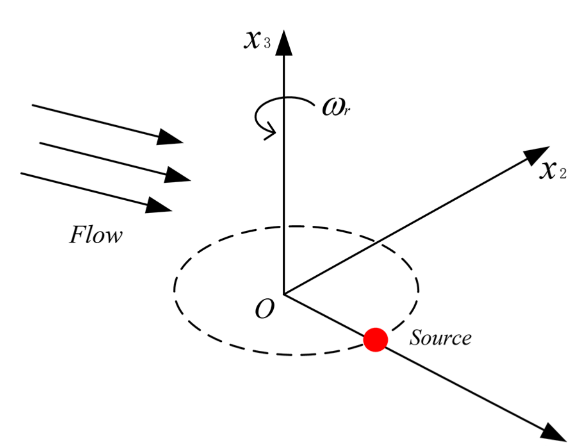

Fig. 3 Schematic of a rotating monopole in a moving medium.

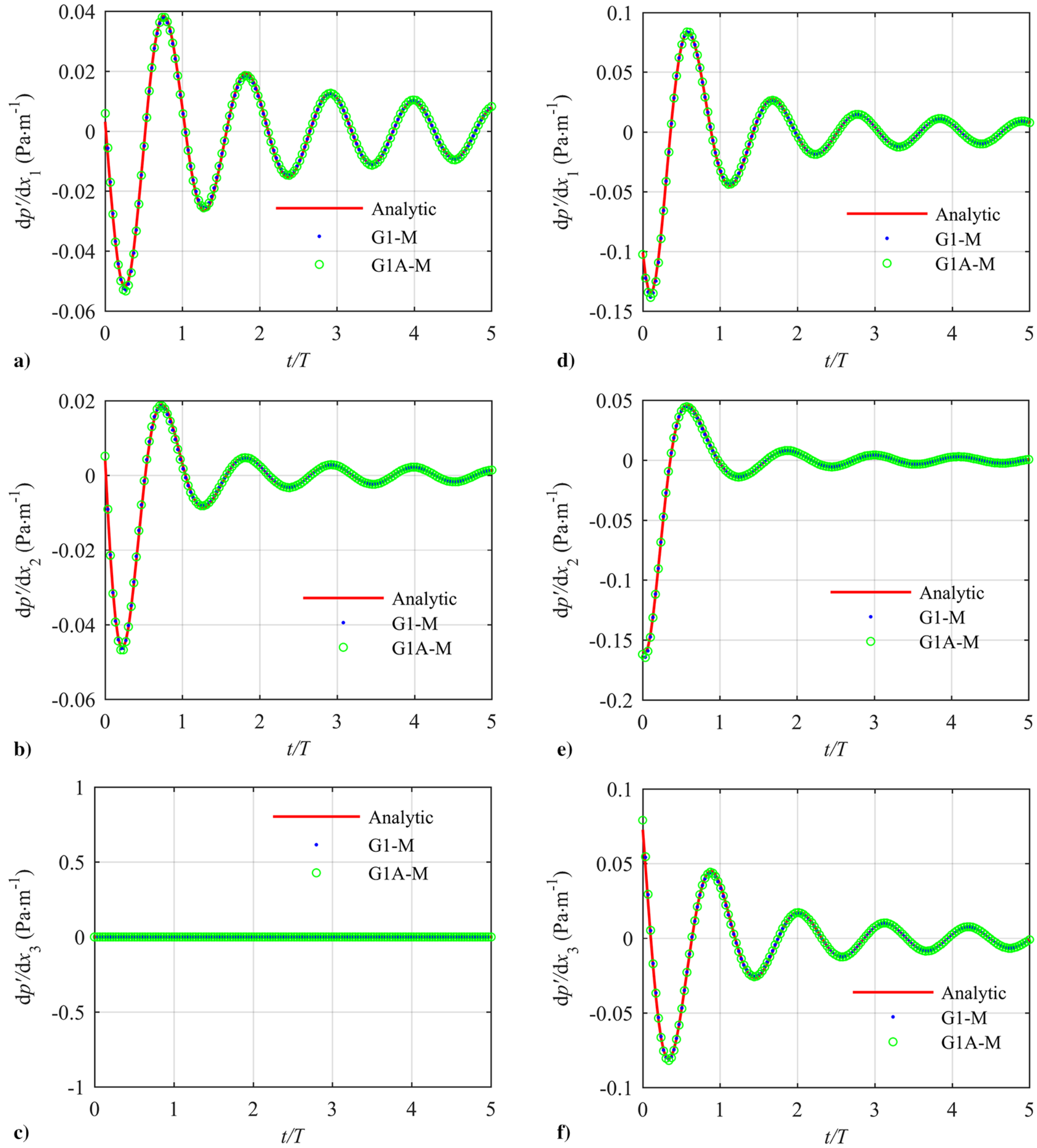

Fig. 4 Acoustic pressure gradient time history of a rotating monopole: a–c) $M_\infty = (0,0,0)$; d–f) $M_\infty = (0.4,0.3,0.5)$.

The excellent agreement further validates the reliability and accuracy of proposed formulations G1-M and G1A-M for acoustic pressure gradient prediction in the time domain. Because of the aforementioned reasons, the computation time for formulation G1-M of 3.505 s is longer than that for formulation G1A-M of 1.541 s.

C. Test Case 3: Rotating Monopole Source in a Moving Medium

To show the feasibility and applicability of the proposed formulations for a moving source in a moving medium, a rotating monopole case is considered, as shown in Fig. 3.

The monopole rotates counterclockwise around the x_3 axis with an angular speed of $\omega_r = 2\pi$ rad/s at a radius of $rs = 1$ m in the x_1x_2 plane, and the initial position is $\boldsymbol{x}_s = (1, 0, 0)$m when $\tau = 0$. Corresponding parameters A, ω, c_0, and ρ_0, as well as the time sampling points in one source period, are set the same values as those in the stationary monopole case. In this case, a spherical data surface, which is the same as the one used in the previous two test cases enclosing the monopole moving along with the source, is adopted to predict the acoustic pressure gradient time history at the observer.

The observer is moving along the x_1 axis at a constant velocity of $v_o = 30$ m/s, and its initial position is $\boldsymbol{x}_o = (5, 5, 0)$m. In this test case, there are two flow Mach numbers considered, corresponding to a medium at rest and a moving medium of $\boldsymbol{M}_\infty = (0.4, 0.3, 0.5)$. Figure 4 depicts the acoustic pressure gradient time history predicted by formulations G1-M and G1A-M. The excellent agreement between the predicted results and the analytic solutions proves the capability of the derived formulations to predict accurately the acoustic pressure gradient in a moving medium. Here, the computation time for formulation G1-M is 9.410 s and that for formulation G1A-M is 3.947 s.

IV. Conclusions

In this paper, based on the convective FW–H equation, both semianalytic time-domain formulation G1-M and analytic time-domain formulation G1A-M for the prediction of the acoustic pressure gradient in a moving medium were derived. Although a moving medium (for example, in a wind tunnel) can be equivalently solved in a stationary medium as well by using a moving observer, which was justified in [23], formulations G1-M and G1A-M (which explicitly take into account the presence of the uniform flow) are more easily interpreted to examine the convective effects.

The validity and applicability of the derived formulations were verified through three computational test cases consisting of a stationary monopole source, a stationary dipole source, as well as a rotating monopole source. Different flow configurations were considered to obtain the predicted acoustic pressure gradient data, and the agreement between the predicted results and analytic solutions was excellent. Meanwhile, the computational efficiencies of formulations G1-M and G1A-M to deal with the moving observer cases were compared, and it was found that formulation G1A-M led to a more efficient calculation than formulation G1-M because it eliminated the observer time differentiation of the integrals.

Derived formulations G1-M and G1A-M explicitly take into account the presence of the moving medium, and thus can be used to predict the acoustic pressure gradient on the scattering surface, which can serve as the boundary condition in the aeroacoustic scattering calculation. In future work, the authors will consider aeroacoustic scattering phenomena in the time domain using these formulations.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (grant nos. 11674082 and 51405125). The authors want to acknowledge Earl H. Dowell of Duke University for his help in revising the manuscript to improve its readability. The authors also want to acknowledge Seongkyu Lee of University of California, Davis (the author of the original formulations of G1 and G1A) for permitting us to append an extension (−M) to formulations G1 and G1A.

References

[1] Hanson, D. B., and Magliozzi, B., "Propagation of Propeller Tone Noise Through a Fuselage Boundary Layer," *Journal of Aircraft*, Vol. 22, No. 1, 1985, pp. 63–70.
doi:10.2514/3.45081

[2] Atalla, N., and Glegg, S., "A Ray-Acoustics Approach to Fuselage Scattering of Rotor Noise," *13th Aeroacoustics Conference*, AIAA Paper 1990-4013, 1990.
doi:10.2514/6.1990-4013

[3] Atalla, N., and Glegg, S., "Ray-Acoustics Approach to Fuselage Scattering of Rotor Noise," *Journal of the American Helicopter Society*, Vol. 38, No. 3, 1993, pp. 56–63.
doi:10.4050/JAHS.38.56

[4] Hu, F., "An Efficient Solution of Time Domain Boundary Integral Equations for Acoustic Scattering and Its Acceleration by Graphics Processing Units," *19th AIAA/CEAS Aeroacoustics Conference*, AIAA Paper 2013-2018, 2013.
doi:10.2514/6.2013-2018

[5] Lee, S., Erwin, J. P., and Brentner, K. S., "A Method to Predict Acoustic Scattering of Rotorcraft Noise," *Journal of the American Helicopter Society*, Vol. 54, No. 4, 2009, Paper 042007.
doi:10.4050/JAHS.54.042007

[6] Lee, S., Brentner, K. S., and Morris, P. J., "Time-Domain Approach for Acoustic Scattering of Rotorcraft Noise," *Journal of the American Helicopter Society*, Vol. 57, No. 4, 2012, Paper 042001.
doi:10.4050/JAHS.57.042001

[7] Glegg, S. A. L., "Effect of Centerbody Scattering on Propeller Noise," *AIAA Journal*, Vol. 29, No. 4, 1991, pp. 572–576.
doi:10.2514/3.10622

[8] Kingan, M. J., and Self, R. H., "Open Rotor Tone Scattering," *Journal of Sound and Vibration*, Vol. 331, No. 8, 2012, pp. 1806–1828.
doi:10.1016/j.jsv.2011.12.001

[9] Kingan, M. J., and Sureshkumar, P., "Open Rotor Centrebody Scattering," *Journal of Sound and Vibration*, Vol. 333, No. 2, 2014, pp. 418–433.
doi:10.1016/j.jsv.2013.08.010

[10] Mao, Y., and Qi, D., "Computation of Rotating Blade Noise Scattered by a Centrifugal Volute," *Journal of Power and Energy*, Vol. 223, No. A8, 2009, pp. 965–972.
doi:10.1243/09576509JPE794

[11] Crighton, D. G., and Leppington, F. G., "On the Scattering of Aerodynamic Noise," *Journal of Fluid Mechanics*, Vol. 46, No. 3, 1971, pp. 577–597.
doi:10.1017/S0022112071000715

[12] Ffowcs Williams, J. E., and Hawkings, D. L., "Sound Generation by Turbulence and Surfaces in Arbitrary Motion," *Philosophical Transactions of the Royal Society of London, Series A: Mathematical, Physical and Engineering Sciences*, Vol. 264, No. 1151, 1969, pp. 321–342.
doi:10.1098/rsta.1969.0031

[13] Farassat, F., and Myers, M. K., "Extension of Kirchhoff's Formula to Radiation from Moving Surfaces," *Journal of Sound and Vibration*, Vol. 123, No. 3, 1988, pp. 451–460.
doi:10.1016/S0022-460X(88)80162-7

[14] Ergin, A. A., Shanker, B., and Michielssen, E., "Analysis of Transient Wave Scattering from Rigid Bodies Using a Burton-Miller Approach," *Journal of the Acoustical Society of America*, Vol. 106, No. 5, 1999, pp. 2396–2404.
doi:10.1121/1.428076

[15] Testa, C., Ianniello, S., Bernardini, G., and Gennaretti, M., "Sound Scattered by a Helicopter Fuselage in Descent Flight Condition," *13th AIAA/CEAS Aeroacoustics Conference*, AIAA Paper 2007-3497, 2007.
doi:10.2514/6.2007-3497

[16] Dunn, M. H., and Tinetti, A. F., "Aeroacoustic Scattering via the Equivalent Source Method," *10th AIAA/CEAS Aeroacoustics Conference*, AIAA Paper 2004-2937, 2004.
doi:10.2514/6.2004-2937

[17] Lee, S., Brentner, K. S., and Morris, P. J., "Acoustic Scattering in the Time Domain Using an Equivalent Source Method," *AIAA Journal*, Vol. 48, No. 12, 2010, pp. 2772–2780.
doi:10.2514/1.45132

[18] Lee, S., Brentner, K. S., and Morris, P. J., "Assessment of Time-Domain Equivalent Source Method for Acoustic Scattering," *AIAA Journal*, Vol. 49, No. 9, 2011, pp. 1897–1906.
doi:10.2514/1.J050736

[19] Ghorbaniasl, G., Carley, M., and Lacor, C., "Acoustic Velocity Formulation for Sources in Arbitrary Motion," *AIAA Journal*, Vol. 51, No. 3, 2013, pp. 632–642.
doi:10.2514/1.J051958

[20] Mao, Y., Zhang, Q., Xu, C., and Qi, D., "Two Types of Frequency-Domain Acoustic-Velocity Formulations for Rotating Thickness and Loading Sources," *AIAA Journal*, Vol. 53, No. 3, 2015, pp. 713–722.
doi:10.2514/1.J053230

[21] Lee, S., and Brentner, K. S., "Comment on "Acoustic Velocity Formulation for Sources in Arbitrary Motion," *AIAA Journal*, Vol. 54, No. 5, 2016, pp. 1810–1811.
doi:10.2514/1.J054845

[22] Farassat, F., and Brentner, K. S., "The Derivation of the Gradient of the Acoustic Pressure on a Moving Surface for Application to the Fast Scattering Code (FSC)," NASA TM-2005-213777, 2005.

[23] Lee, S., Brentner, K. S., Farassat, F., and Morris, P. J., "Analytic Formulation and Numerical Implementation of an Acoustic Pressure Gradient Prediction," *Journal of Sound and Vibration*, Vol. 319, Nos. 3–5, 2009, pp. 1200–1221.
doi:10.1016/j.jsv.2008.06.028

[24] Wells, V. L., and Han, A. Y., "Acoustics of a Moving Source in a Moving Medium with Application to Propeller Noise," *Journal of Sound and Vibration*, Vol. 184, No. 4, 1995, pp. 651–663.
doi:10.1006/jsvi.1995.0339

[25] Xu, C., Mao, Y., and Qi, D., "Frequency-Domain Acoustic Pressure Formulation for Rotating Source in Uniform Subsonic Inflow with Arbitrary Direction," *Journal of Sound and Vibration*, Vol. 333, No. 14, 2014, pp. 3081–3091.
doi:10.1016/j.jsv.2014.03.004

[26] Ghorbaniasl, G., Huang, Z., Siozos-Rousoulis, L., and Lacor, C., "Analytical Acoustic Pressure Gradient Prediction for Moving Medium Problems," *Proceedings of the Royal Society of London, Series A: Mathematical, Physical and Engineering Sciences*, Vol. 471, No. 2184, 2015, pp. 1–14.
doi:10.1098/rspa.2015.0342

[27] Ghorbaniasl, G., and Lacor, C., "A Moving Medium Formulation for Prediction of Propeller Noise at Incidence," *Journal of Sound and Vibration*, Vol. 331, No. 1, 2012, pp. 117–137.
doi:10.1016/j.jsv.2011.08.018

[28] Farassat, F., "Introduction to Generalized Functions with Applications in Aerodynamics and Aeroacoustics," NASA TP-3428, April 1996.

[29] Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P., *Numerical Recipes: The Art of Scientific Computing*, 3rd ed., Cambridge Univ. Press, New York, 2007, Chap. 5.

D. Papamoschou
Associate Editor