```
> restart; read "/Users/heba/Desktop/Implementations/Reduce_Order_Algorithm.txt" :
> with(LREtools) :
  _Env_LRE_tau := tau;
  _Env_LRE_x := x;
```

$$\_Env\_LRE\_tau := \tau$$

$$\_Env\_LRE\_x := x \quad (1)$$

```
> read "/Users/heba/Desktop/Implementations/Conic.txt" :
>
> #OEIS example A295371:

> L3 := (2*x+1)*(x+3)^2*tau^3-(2*x+1)*(7*x^2+38*x+52)*tau^2-3*(2*x
      +5)*(7*x^2+4*x+1)*tau+27*(2*x+5)*x^2;
```

$$L3 := (2x+1)(x+3)^2\tau^3 - (2x+1)\left(7x^2+38x+52\right)\tau^2 - 3(2x+5)\left(7x^2+4x+1\right)\tau + 27(2x+5)x^2 \quad (2)$$

```
> G, Ginv, L2, r := ReduceOrder(L3);
```

$$G, Ginv, L2, r := \left(x^4+5x^3+9x^2+8x+4\right)\tau^2 - \frac{2\left(4x^5+21x^4+42x^3+44x^2+27x+6\right)\tau}{2x+1} + \frac{3x\left(2x^4+9x^3+15x^2+13x+6\right)}{2x+1}, -\frac{(2x+1)^2(x+1)\tau^2}{72x\left(4x^2+20x+21\right)\left(x^2+3x+3\right)} + \frac{\left(48x^7+480x^6+1952x^5+4236x^4+5373x^3+4027x^2+1650x+270\right)\tau}{36(2x+3)\left(x^2+x+1\right)\left(4x^2+20x+21\right)\left(x^2+3x+3\right)x(2x+5)} + \frac{18x^3+45x^2+31x+8}{24(2x+3)\left(x^2+x+1\right)(x+2)x}, \tau^2+\tau-\frac{3x}{4(x+2)}, \frac{4(x+1)(2x+5)\left(x^2+3x+3\right)}{\left(x^2+x+1\right)(2x+3)x} \quad (3)$$

```
> L2;
```

$$\tau^2+\tau-\frac{3x}{4(x+2)} \quad (4)$$

```
> #Using Giles Levy's implementation to find second order operator of an OEIS entry that is gauge
      equivalent to L2:

> read "/Users/heba/Desktop/Implementations/Giles_Levy_implementation/code/findrel_v2.7.txt" :
> _Env_LRE_tau := tau;
  _Env_LRE_x := x;
```

$$\_Env\_LRE\_tau := \tau$$

$$\_Env\_LRE\_x := x \quad (5)$$

```
> _tau := tau; LREtools[OperatorToRecurrence](L2, u(n)); numer(lhs(%));
```

$$\_\tau := \tau$$

$$u(n+2) + u(n+1) - \frac{3\,n\,u(n)}{4\,(n+2)} = 0$$

$$4\,u(n+2)\,n + 4\,u(n+1)\,n - 3\,n\,u(n) + 8\,u(n+2) + 8\,u(n+1) \tag{6}$$

```
> findrel(%, u(n));
```

Warning, not all solutions found

$$u(n) = \_c\left(\frac{\left(-\frac{1}{2}\right)^n (n+2)\,(n-3)\,A001006(n)}{n} + \frac{\left(-\frac{1}{2}\right)^n (n+2)\,(n+3)\,A001006(n+1)}{n}\right) \tag{7}$$

```
> #Finding gauge transformation between L3 and the symmetric square of L_A001006:
> read "/Users/heba/Desktop/Implementations/ProjHom/Applications";
```

$$\_Env\_LRE\_tau := \tau$$

$$\_Env\_LRE\_x := x \tag{8}$$

```
> # L2a = A001006^2
  L2a := (x + 4) * (2 * x + 5) * (x + 5)^2 * tau^3 - (x + 4) * (2 * x + 7) * (7 * x^2 + 42
      * x + 59) * tau^2 - 3 * (2 * x + 5) * (x + 2) * (7 * x^2 + 42 * x + 59) * tau + 27 * (2
      * x + 7) * (x + 2) * (x + 1)^2;
```

$$L2a := (x+4)\,(2\,x+5)\,(x+5)^2\,\tau^3 - (x+4)\,(2\,x+7)\,\left(7\,x^2+42\,x+59\right)\tau^2 - 3\,(2\,x+5)\,(x+2)\,\left(7\,x^2+42\,x+59\right)\tau + 27\,(2\,x+7)\,(x+2)\,(x+1)^2 \tag{9}$$

```
> # L3 = A295371
  L3 := (2 * x + 1) * (x + 3)^2 * tau^3 - (2 * x + 1) * (7 * x^2 + 38 * x + 52) * tau^2 - 3 * (2 * x
      + 5) * (7 * x^2 + 4 * x + 1) * tau + 27 * (2 * x + 5) * x^2;
```

$$L3 := (2\,x+1)\,(x+3)^2\,\tau^3 - (2\,x+1)\,\left(7\,x^2+38\,x+52\right)\tau^2 - 3\,(2\,x+5)\,\left(7\,x^2+4\,x+1\right)\tau + 27\,(2\,x+5)\,x^2 \tag{10}$$

```
> G := ProjectiveHom(L2a, subs(x = x + 1, L3));
```

PairsAB: number of combinations left after comparing with local data
c,s,d 15
deltaHS: Computed 5 ABPairs 3.248

$$G := SolOf\left(\tau - \frac{(2\,x+5)\,(x+1)}{(2\,x+7)\,(x+2)}\right)\left((2\,x+3)\,(x+3)\,(x+4)^2\,\tau^2 - (2\,x+5)\,(x+3)\,\left(7\,x^2+28\,x+24\right)\tau - 12\,\left(5\,x^3+29\,x^2+54\,x+33\right)(x+1)\right) \tag{11}$$

```
> op(1, G);
```

$$SolOf\left(\tau - \frac{(2\,x+5)\,(x+1)}{(2\,x+7)\,(x+2)}\right) \tag{12}$$

```
> op(%);
```

$$\tau - \frac{(2\,x+5)\,(x+1)}{(2\,x+7)\,(x+2)} \tag{13}$$

```
> LREtools[OperatorToRecurrence](%, u(n));
```

$$u(n+1) - \frac{(2n+5)(n+1)u(n)}{(2n+7)(n+2)} = 0 \quad (14)$$

```
> LREtools[hypergeomsols](%, u(n), { }, output = basis);
```

$$\left[\frac{1}{(2n+5)(n+1)}\right] \quad (15)$$

$$> \frac{G}{op(1,G)} \cdot \%[1];$$

$$\frac{1}{(2n+5)(n+1)}\Big((2x+3)(x+3)(x+4)^2\tau^2 - (2x+5)(x+3)\left(7x^2+28x+24\right)\tau - 12\left(5x^3+29x^2+54x+33\right)(x+1)\Big) \quad (16)$$

```
> G := subs(n = x, %); collect(%, tau, factor);
```

$$G := \frac{1}{(2x+5)(x+1)}\Big((2x+3)(x+3)(x+4)^2\tau^2 - (2x+5)(x+3)\left(7x^2+28x+24\right)\tau - 12\left(5x^3+29x^2+54x+33\right)(x+1)\Big)$$

$$\frac{(2x+3)(x+3)(x+4)^2\tau^2}{(2x+5)(x+1)} - \frac{(x+3)\left(7x^2+28x+24\right)\tau}{x+1} - \frac{12\left(5x^3+29x^2+54x+33\right)}{2x+5} \quad (17)$$

```
> # Copied from OEIS A001006:
  sq := [1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511,
   41835, 113634, 310572, 853467, 2356779, 6536382,
   18199284, 50852019, 142547559, 400763223, 1129760415,
   3192727797, 9043402501, 25669818476, 73007772802,
   208023278209, 593742784829, 1697385471211];
```

sq := [1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, 2356779, 6536382, 18199284, 50852019, 142547559, 400763223, 1129760415, 3192727797, 9043402501, 25669818476, 73007772802, 208023278209, 593742784829, 1697385471211] **(18)**

```
> seq(add(eval(coeff(G, tau, j), x = i) * sq[1 + i + j]^2, j = 0..2), i = 0..10);
```

−36, −108, −684, −4572, −33156, −249156, −1926828, −15199164, −121767012, −987445548, −8086905324 **(19)**

```
> #This is -36·A295371, proving that 36·A295371 has integer entries
>
>
>
> #Next, use another OEIS entry from giles "groups" file, namely A002426.
> #Then 4·A295371 is integer sequence.
> # Then read that entry to get data mod 2.
>
```