```
> #OEIS A268138:

> restart; read "/Users/heba/Desktop/Implementations/Reduce_Order_Algorithm.txt" :
> read "/Users/heba/Desktop/Implementations/Conic.txt" :
> symmprod := proc(L1, L2, τ) local n1, n2, sb1, i, sb2, P, var, j, IsZero, EQN, s, NumberFree, k,
     L;

   n1 := degree(L1, τ);
   n2 := degree(L2, τ);
   sb1 := u(x + n1) = -add(coeff(L1, τ, i)/lcoeff(L1, τ) · u(x + i), i = 0 ..n1 − 1);

   sb2 := v(x + n2) = -add(coeff(L2, τ, i)/lcoeff(L2, τ) · v(x + i), i = 0 ..n2 − 1);
   P[0] := u(x)·v(x);
   for i to n1·n2 do
    P[i] := subs(sb1, sb2, subs(x = x + 1, P[i-1]))
   od;
  var := {seq(u(x + i), i = 0 ..n1-1), seq(v(x + j), j = 0 ..n2-1)};

  IsZero := collect(add(c[i]·P[i], i = 0 .. n1·n2), var, distributed);
  EQN := {coeffs(IsZero, var)};
   var := {seq(c[i], i = 0 ..n1·n2)};
  s := solve(EQN, var);
   NumberFree := add(`if`(lhs(i) = rhs(i), 1, 0), i = s);
   if NumberFree > 1 then
    s := solve(s union {seq(c[k] = 0, k = n1·n2 + 2 − NumberFree ... n1·n2)}, var)
   fi;
   L := sort(collect(primpart(subs(s, add(c[i]·τ^i, i = 0 ..n1·n2)), τ), tau, factor), tau);

   end proc
symmprod := proc(L1, L2, tau)                                                    (1)
    local n1, n2, sb1, i, sb2, P, var, j, IsZero, EQN, s, NumberFree, k, L;
    n1 := degree(L1, tau);
    n2 := degree(L2, tau);
    sb1 := u(x + n1) = − add(coeff(L1, tau, i) * u(x + i) / lcoeff(L1, tau), i = 0 ..n1 − 1);
    sb2 := v(x + n2) = − add(coeff(L2, tau, i) * v(x + i) / lcoeff(L2, tau), i = 0 ..n2 − 1);
    P[0] := u(x) * v(x);
    for i to n1 * n2 do P[i] := subs(sb1, sb2, subs(x = x + 1, P[i − 1])) end do;
    var := {seq(u(x + i), i = 0 ..n1 − 1), seq(v(x + j), j = 0 ..n2 − 1)};
    IsZero := collect(add(c[i] * P[i], i = 0 ..n1 * n2), var, distributed);
    EQN := {coeffs(IsZero, var)};
```

```
    var := {seq(c[i], i = 0 ..n1 * n2)};
    s := solve(EQN, var);
    NumberFree := add(if(lhs(i) = rhs(i), 1, 0), i = s);
    if 1 < NumberFree then
        s := solve(s union {seq(c[k] = 0, k = n1 * n2 + 2 − NumberFree ..n1 * n2)}, var)
    end if;
    L := sort(collect(primpart(subs(s, add(c[i] * tau^i, i = 0 ..n1 * n2))), tau), tau, factor), tau)
end proc
```

```
> symmprod((x + 2)·τ^2 − 3·(2·x + 3)·τ + (x + 1), subs(x = x + 1, (x + 3)·τ^2 − (6·x + 9)·τ
        + x), tau);
```

$$(2x+5)(x+6)(x+5)(x+4)\tau^4-(x+5)\left(72x^3+752x^2+2549x+2796\right)\tau^3+\left(140x^4+1750x^3+8051x^2+16157x+11948\right)\tau^2-(x+2)\left(72x^3+544x^2+1299x+971\right)\tau+(2x+7)(x+2)(x+1)^2 \quad \textbf{(2)}$$

```
> L := LREtools[MultiplyOperators](%, τ − 1) ;
```

$$L := \left(2\tau^4x^4+35\tau^4x^3-72\tau^3x^4+223\tau^4x^2-1112\tau^3x^3+140\tau^2x^4+610\tau^4x-6309\tau^3x^2+1750\tau^2x^3-72\tau x^4+600\tau^4-15541\tau^3x+8051\tau^2x^2-688\tau x^3+2x^4-13980\tau^3+16157\tau^2x-2387\tau x^2+15x^3+11948\tau^2-3569\tau x+38x^2-1942\tau+39x+14\right)(\tau-1) \quad \textbf{(3)}$$

```
> L3 := symmprod(%, tau − x/(x + 1), tau);
```

$$L3 := -(2x+5)(x+4)(x+6)(x+5)^2\tau^5+(x+5)(x+4)\left(74x^3+777x^2+2647x+2916\right)\tau^4-2(x+4)(x+3)(2x+7)\left(53x^2+318x+463\right)\tau^3+2(2x+5)(x+3)(x+2)\left(53x^2+318x+463\right)\tau^2-(x+2)(x+1)\left(74x^3+555x^2+1315x+978\right)\tau+x(2x+7)(x+2)(x+1)^2 \quad \textbf{(4)}$$

```
> with(LREtools) : _Env_LRE_tau := tau; _Env_LRE_x := x
```

$$\_Env\_LRE\_tau := \tau$$

$$\_Env\_LRE\_x := x \quad \textbf{(5)}$$

```
> L3 := LREtools[OperatorToRecurrence](L3, u(n));
```

$$L3 := -(2n+5)(n+4)(n+6)(n+5)^2u(n+5)+(n+5)(n+4)\left(74n^3+777n^2+2647n+2916\right)u(n+4)-2(n+4)(n+3)(2n+7)\left(53n^2+318n+463\right)u(n+3)+2(2n+5)(n+3)(n+2)\left(53n^2+318n+463\right)u(n+2)-(n+2)(n+1)\left(74n^3+555n^2+1315n+978\right)u(n+1)+n(2n+7)(n+2)(n+1)^2u(n)=0 \quad \textbf{(6)}$$

```
> LREtools[MinimalRecurrence](L3, u(n), {u(1) = 1, u(2) = 5, u(3) = 51, u(4) = 747, u(5)
```

$= 13245\});$

$$(n+4)\,(2n+3)\,(n+3)^2\,u(n+3) - (2n+5)\,(n+3)\,(35n^2+107n+82)\,u(n+2) + (2n+3)\,(n+1)\,(35n^2+173n+214)\,u(n+1) - (2n+5)\,(n+1)^2\,n\,u(n) = 0, \text{"Relation holds for"}, 1 \le n, \text{"Initial terms"}, \{u(1)=1, u(2)=5, u(3)=51\} \quad \textbf{(7)}$$

> %[1];

$$(n+4)\,(2n+3)\,(n+3)^2\,u(n+3) - (2n+5)\,(n+3)\,(35n^2+107n+82)\,u(n+2) + (2n+3)\,(n+1)\,(35n^2+173n+214)\,u(n+1) - (2n+5)\,(n+1)^2\,n\,u(n) = 0 \quad \textbf{(8)}$$

> L3 := LREtools[RecurrenceToOperator](%, u(n));

$$L3 := (x+4)\,(2x+3)\,(x+3)^2\,\tau^3 - (2x+5)\,(x+3)\,(35x^2+107x+82)\,\tau^2 + (2x+3)\,(x+1)\,(35x^2+173x+214)\,\tau - (2x+5)\,(x+1)^2\,x \quad \textbf{(9)}$$

> G, Ginv, L2, r := ReduceOrder(L3);

$$G, Ginv, L2, r := (64x^5+512x^4+1490x^3+1918x^2+1056x+216)\,\tau^2 + (-1152x^5 - 6080x^4 - 11844x^3 - 10666x^2 - 4542x - 756)\,\tau + 1088x^5 + 4544x^4 + 6514x^3 + 3724x^2 + 774x,$$
$$-\frac{(12x^2+24x+11)\,(x^2+5x+6)\,(x+3)\,(2x+3)\,\tau^2}{96\,(8x^3+68x^2+190x+175)\,x\,(6x^3+30x^2+47x+23)\,(32x^2+96x+73)} + \big((13056x^9+195840x^8+1266008x^7+4632536x^6+10582084x^5+15665613x^4+15056628x^3+9086761x^2+3137562x+474039)\,\tau\big) \big/ \big(48\,(32x^3+96x^2+73x+18)\,(4x^2+24x+35)\,(32x^2+96x+73)\,(6x^3+30x^2+47x+23)\,x\,(2x+3)\big) - \frac{12x^4+72x^3+219x^2+389x+270}{96\,(32x^2+32x+9)\,(6x^2+24x+23)\,x\,(4x^2+12x+9)}, \tau^2+\tau + \frac{(x+2)\,(2x+1)\,(x+1)\,(2x+5)}{4\,(6x^2+24x+23)\,(6x^2+12x+5)},$$
$$\frac{4\,(32x^2+96x+73)\,(2x+5)\,(6x^2+12x+5)^2}{(4x^2+4x+1)\,(32x^2+32x+9)\,(x+2)^2\,(2x+3)} \quad \textbf{(10)}$$

> L2;

$$\tau^2+\tau+\frac{(x+2)\,(2x+1)\,(x+1)\,(2x+5)}{4\,(6x^2+24x+23)\,(6x^2+12x+5)} \quad \textbf{(11)}$$

>

#Using Giles Levy's implementation to find second order operator of an OEIS entry that is gauge equivalent to L2:

> **read** "/Users/heba/Desktop/Implementations/Giles_Levy_implementation/code/findrel_v2.7.txt" :

> _Env_LRE_tau := tau;
_Env_LRE_x := x;

$$\_Env\_LRE\_tau := \tau$$
$$\_Env\_LRE\_x := x \quad \textbf{(12)}$$

```
> _tau := tau; LREtools[OperatorToRecurrence](L2, u(n)); numer(lhs(%));
```

$$\_\tau := \tau$$

$$u(n+2) + u(n+1) + \frac{(n+2)\,(2\,n+1)\,(n+1)\,(2\,n+5)\,u(n)}{4\,\left(6\,n^2+24\,n+23\right)\,\left(6\,n^2+12\,n+5\right)} = 0$$

$$144\,u(n+2)\,n^4 + 144\,u(n+1)\,n^4 + 4\,u(n)\,n^4 + 864\,u(n+2)\,n^3 + 864\,u(n+1)\,n^3 + 24\,u(n)\,n^3 + 1824\,u(n+2)\,n^2 + 1824\,u(n+1)\,n^2 + 49\,u(n)\,n^2 + 1584\,u(n+2)\,n + 1584\,u(n+1)\,n + 39\,u(n)\,n + 460\,u(n+2) + 460\,u(n+1) + 10\,u(n) \tag{13}$$

```
> findrel(%, u(n));
Warning, not all solutions found
```

$$u(n) = \_c\left(\frac{\left(-\frac{1}{6}\right)^n\,\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)\,n^2\,(n+1)\,A001003(n)}{\Gamma\left(n-\frac{\sqrt{6}}{6}\right)\,\Gamma\left(n+\frac{\sqrt{6}}{6}\right)\,\left(6\,n^2-1\right)} + \frac{\Gamma(n)\,\left(-\frac{1}{6}\right)^n\,\Gamma\left(n+\frac{1}{2}\right)\,(n+2)\,(n+1)\,n\,A001003(n+1)}{\Gamma\left(n-\frac{\sqrt{6}}{6}\right)\,\Gamma\left(n+\frac{\sqrt{6}}{6}\right)\,\left(6\,n^2-1\right)}\right) \tag{14}$$

```
> #Finding gauge transformation between L3 and the symmetric square of L_A001003:
> read "/Users/heba/Desktop/Implementations/ProjHom/Applications";
```

$$\_Env\_LRE\_tau := \tau$$

$$\_Env\_LRE\_x := x \tag{15}$$

```
> # L = A001003^2
> L := symmprod(((x+3)·τ^2 − (6·x+9)·τ + x)$2, tau);
```

$$L := -(x+3)\,(2\,x+3)\,(x+4)^2\,\tau^3 + (2\,x+5)\,(x+3)\,\left(35\,x^2+140\,x+132\right)\,\tau^2 - (2\,x+3)\,(x+1)\,\left(35\,x^2+140\,x+132\right)\,\tau + (2\,x+5)\,x^2\,(x+1) \tag{16}$$

```
> # L3 = A268138
  L3;
```

$$(x+4)\,(2\,x+3)\,(x+3)^2\,\tau^3 - (2\,x+5)\,(x+3)\,\left(35\,x^2+107\,x+82\right)\,\tau^2 + (2\,x+3)\,(x+1)\,\left(35\,x^2+173\,x+214\right)\,\tau - (2\,x+5)\,(x+1)^2\,x \tag{17}$$

```
> G := ProjectiveHom(L, subs(x=x+1, L3));
PairsAB: number of combinations left after comparing with local data
c,s,d 3
deltaHS: Computed 1 ABPairs 3.937
```

$$G := SolOf(\tau-1)\,\left((x+3)^2\,\tau^2 + \left(-34\,x^2-102\,x-77\right)\,\tau + x^2\right) \tag{18}$$

```
> op(1, G);
```

$$SolOf(\tau-1) \tag{19}$$

```
> op(%);
```

$$\tau - 1 \tag{20}$$

```
> LREtools[OperatorToRecurrence](%, u(n));
```
$$u(n+1) - u(n) = 0 \quad (21)$$

```
> LREtools[hypergeomsols](%, u(n), { }, output = basis);
```
$$[1] \quad (22)$$

```
> G/op(1, G) · %[1];
```
$$(x+3)^2 \tau^2 + \left(-34 x^2 - 102 x - 77\right) \tau + x^2 \quad (23)$$

```
> G := subs(n = x, %); collect(%, tau, factor);
```
$$G := (x+3)^2 \tau^2 + \left(-34 x^2 - 102 x - 77\right) \tau + x^2$$
$$(x+3)^2 \tau^2 + \left(-34 x^2 - 102 x - 77\right) \tau + x^2 \quad (24)$$

```
> #A001003:
  sq := [1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859,
   2646723, 13648869, 71039373, 372693519, 1968801519,
   10463578353, 55909013009, 300159426963,
   1618362158587, 8759309660445, 47574827600981,
   259215937709463, 1416461675464871]
```
sq := [1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, 2646723, 13648869, 71039373, 372693519, 1968801519, 10463578353, 55909013009, 300159426963, 1618362158587, 8759309660445, 47574827600981, 259215937709463, 1416461675464871] **(25)**

```
> seq(add(eval(coeff(G, tau, j), x = i) * sq[1 + i + j]^2, j = 0 ..2), i = 0 ..10);
```
4, 20, 204, 2988, 52980, 1057316, 22885660, 525700316, 12637559268, 314919393588, 8079641301996 **(26)**

```
> #Let c(n) be the OEIS entry A001003. Using the recurrence of A001003, we write c(n+2) in terms
     of c(n+1) and c(n). Substituting c(n+2) in G, gives us the formula in Example 5.3.3 in the
     paper.
```