```
restart;
> #OEIS A178808:
> restart; read "/Users/heba/Desktop/Implementations/Reduce_Order_Algorithm.txt" :
> read "/Users/heba/Desktop/Implementations/Conic.txt" :
> symmprod := proc(L1, L2, τ)
  local n1, n2, sb1, i, sb2, P, var, j, IsZero, EQN, s, NumberFree, k, L;
  n1 := degree(L1, τ);
  n2 := degree(L2, τ);
  sb1 := u(x + n1) = -add(coeff(L1, τ, i)/lcoeff(L1, τ) · u(x + i), i = 0 ..n1 - 1);

  sb2 := v(x + n2) = -add(coeff(L2, τ, i)/lcoeff(L2, τ) · v(x + i), i = 0 ..n2 - 1);
  P[0] := u(x)·v(x);
  for i to n1·n2 do
   P[i] := subs(sb1, sb2, subs(x = x + 1, P[i-1]))
  od;
  var := {seq(u(x + i), i = 0 ..n1-1), seq(v(x + j), j = 0 ..n2-1)};

  IsZero := collect(add(c[i]·P[i], i = 0 .. n1·n2), var, distributed);
  EQN := {coeffs(IsZero, var)};
  var := {seq(c[i], i = 0 ..n1·n2)};
  s := solve(EQN, var);
  NumberFree := add(`if`(lhs(i) = rhs(i), 1, 0), i = s);
  if NumberFree > 1 then
   s := solve(s union {seq(c[k] = 0, k = n1·n2 + 2 - NumberFree ... n1·n2)}, var)
  fi;
  L := sort(collect(primpart(subs(s, add(c[i]·τ^i, i = 0 ..n1·n2))), τ), tau, factor), tau);

  end proc
symmprod := proc(L1, L2, tau)                                                   (1)
   local n1, n2, sb1, i, sb2, P, var, j, IsZero, EQN, s, NumberFree, k, L;
   n1 := degree(L1, tau);
   n2 := degree(L2, tau);
   sb1 := u(x + n1) = − add(coeff(L1, tau, i) * u(x + i) / lcoeff(L1, tau), i = 0 ..n1 − 1);
   sb2 := v(x + n2) = − add(coeff(L2, tau, i) * v(x + i) / lcoeff(L2, tau), i = 0 ..n2 − 1);
   P[0] := u(x) * v(x);
   for i to n1 * n2 do P[i] := subs(sb1, sb2, subs(x = x + 1, P[i − 1])) end do;
   var := {seq(u(x + i), i = 0 ..n1 − 1), seq(v(x + j), j = 0 ..n2 − 1)};
   IsZero := collect(add(c[i] * P[i], i = 0 ..n1 * n2), var, distributed);
   EQN := {coeffs(IsZero, var)};
   var := {seq(c[i], i = 0 ..n1 * n2)};
```

```
    s := solve(EQN, var);
    NumberFree := add(if(lhs(i) = rhs(i), 1, 0), i = s);
    if 1 < NumberFree then
        s := solve(s union {seq(c[k] = 0, k = n1*n2 + 2 - NumberFree..n1*n2)}, var)
    end if;
    L := sort(collect(primpart(subs(s, add(c[i]*tau^i, i = 0..n1*n2))), tau), tau, factor), tau)
end proc
```

```
> L := symmprod((x+2)·τ^2 − 3·(2·x+3)·τ + (x+1), (x+2)·τ^2 − 3·(2·x+3)·τ + (x+1), tau);
```

$$L := -(x+3)^2 (2x+3)\tau^3 + (2x+5)\left(35x^2+140x+131\right)\tau^2 - (2x+3)\left(35x^2+140x+131\right)\tau + (2x+5)(x+1)^2 \quad \textbf{(2)}$$

```
>
```

```
> symmprod(L, (τ − (2·x+3)/(2·x+1)), τ);
```

$$(x+3)^2 (2x+3)(2x+1)\tau^3 - (2x+7)(2x+1)\left(35x^2+140x+131\right)\tau^2 + (2x+7)(2x+1)\left(35x^2+140x+131\right)\tau - (2x+7)(2x+5)(x+1)^2 \quad \textbf{(3)}$$

```
> LREtools[MultiplyOperators](%, τ − 1);
```

$$\left(4\tau^3x^4 + 32\tau^3x^3 - 140\tau^2x^4 + 87\tau^3x^2 - 1120\tau^2x^3 + 140\tau x^4 + 90\tau^3x - 3009\tau^2x^2 + 1120\tau x^3 - 4x^4 + 27\tau^3 - 3076\tau^2x + 3009\tau x^2 - 32x^3 - 917\tau^2 + 3076\tau x - 87x^2 + 917\tau - 94x - 35\right)(\tau - 1) \quad \textbf{(4)}$$

```
> L3 := symmprod(%, τ − x^2/(x+1)^2, τ);
```

$$L3 := (2x+3)(2x+1)(x+4)^2(x+3)^2\tau^4 - 2(2x+1)\left(36x^3+270x^2+639x+472\right)(x+3)^2\tau^3 + 2(2x+7)(2x+1)\left(35x^2+140x+131\right)(x+2)^2\tau^2 - 2(2x+7)\left(36x^3+162x^2+207x+68\right)(x+1)^2\tau + x^2(2x+7)(2x+5)(x+1)^2 \quad \textbf{(5)}$$

```
> with(LREtools) : _Env_LRE_tau := tau; _Env_LRE_x := x;
```

$$\_Env\_LRE\_tau := \tau$$

$$\_Env\_LRE\_x := x \quad \textbf{(6)}$$

```
> L3 := LREtools[OperatorToRecurrence](L3, u(n));
```

$$L3 := (2n+3)(2n+1)(n+4)^2(n+3)^2u(n+4) - 2(2n+1)\left(36n^3+270n^2+639n+472\right)(n+3)^2u(n+3) + 2(2n+7)(2n+1)\left(35n^2+140n+131\right)(n+2)^2u(n+2) - 2(2n+7)\left(36n^3+162n^2+207n+68\right)(n+1)^2u(n+1) + n^2(2n+7)(2n+5)(n+1)^2u(n) = 0 \quad \textbf{(7)}$$

```
> LREtools[MinimalRecurrence](L3, u(n), {u(1) = 1, u(2) = 7, u(3) = 97, u(4) = 1791});
```

$$(n+2)(2n+1)(n+3)^2 u(n+3) - (n+2)(2n+1)(35n^2+141n+134)u(n+2) + (2n+5)(n+1)(35n^2+69n+26)u(n+1) - (2n+5)(n+1)n^2 u(n) = 0, \text{"Relation holds for"}, 1 \le n, \text{"Initial terms"}, \{u(1)=1, u(2)=7, u(3)=97\} \quad \textbf{(8)}$$

```
> %[1];
```

$$(n+2)(2n+1)(n+3)^2 u(n+3) - (n+2)(2n+1)(35n^2+141n+134)u(n+2) + (2n+5)(n+1)(35n^2+69n+26)u(n+1) - (2n+5)(n+1)n^2 u(n) = 0 \quad \textbf{(9)}$$

```
> L3 := LREtools[RecurrenceToOperator](%, u(n));
```

$$L3 := (x+2)(2x+1)(x+3)^2\tau^3 - (x+2)(2x+1)(35x^2+141x+134)\tau^2 + (2x+5)(x+1)(35x^2+69x+26)\tau - (2x+5)(x+1)x^2 \quad \textbf{(10)}$$

```
> G, Ginv, L2, r := ReduceOrder(L3);
```

$$G, Ginv, L2, r := (2x^2+4x+2)\tau - 2x^2, -\frac{(x+1)(x+2)^2\tau^2}{16x^2(4x^2+16x+15)} + \frac{(17x+26)(x+1)\tau}{16(2x+3)x^2} - \frac{x^3+36x^2+69x+26}{16x^2(4x^2+8x+3)}, \tau^2+\tau+\frac{(x+1)^2}{9(2x+3)(2x+1)}, \frac{9(2x+3)(2x+1)}{(x+1)^2} \quad \textbf{(11)}$$

```
> L2;
```

$$\tau^2+\tau+\frac{(x+1)^2}{9(2x+3)(2x+1)} \quad \textbf{(12)}$$

```
> #Using Giles Levy's implementation to find second order operator of an OEIS entry that is gauge
      equivalent to L2:
>
> read "/Users/heba/Desktop/Implementations/Giles_Levy_implementation/code/findrel_v2.7.txt" :
> _Env_LRE_tau := tau;
  _Env_LRE_x := x;
```

$$\_Env\_LRE\_tau := \tau$$

$$\_Env\_LRE\_x := x \quad \textbf{(13)}$$

```
> _tau := tau; LREtools[OperatorToRecurrence](L2, u(n)); numer(lhs(%));
```

$$\_\tau := \tau$$

$$u(n+2)+u(n+1)+\frac{(n+1)^2 u(n)}{9(2n+3)(2n+1)} = 0$$

$$36u(n+2)n^2 + 36u(n+1)n^2 + u(n)n^2 + 72u(n+2)n + 72u(n+1)n + 2u(n)n + 27u(n+2) + 27u(n+1) + u(n) \quad \textbf{(14)}$$

```
> findrel(%, u(n));
Warning, not all solutions found
```

$$u(n) = 3\ \_c\left(\frac{\left(-\frac{1}{6}\right)^n \Gamma(n)\, n^2\, A001003(n)}{\Gamma\left(n+\frac{1}{2}\right)} - \frac{n\,(n+2)\,\Gamma(n)\left(-\frac{1}{6}\right)^n A001003(n+1)}{3\,\Gamma\left(n+\frac{1}{2}\right)}\right) \quad \textbf{(15)}$$

> *#using website: A001850 := x -> LegendreP(x, 3).*

> *#Finding gauge transformation between L3 and the symmetric square of L_A001850:*

> **read** "/Users/heba/Desktop/Implementations/ProjHom/Applications";

$$\_Env\_LRE\_tau := \tau$$
$$\_Env\_LRE\_x := x \quad \textbf{(16)}$$

> *# L = A001850^2*

> *L*;

$$-(x+3)^2 (2x+3)\,\tau^3 + (2x+5)\left(35x^2+140x+131\right)\tau^2 - (2x+3)\left(35x^2+140x+131\right)\tau + (2x+5)(x+1)^2 \quad \textbf{(17)}$$

> *# L3 = A178808*
> *L3*;

$$(x+2)(2x+1)(x+3)^2\tau^3 - (x+2)(2x+1)\left(35x^2+141x+134\right)\tau^2 + (2x+5)(x+1)\left(35x^2+69x+26\right)\tau - (2x+5)(x+1)x^2 \quad \textbf{(18)}$$

> *G* := *ProjectiveHom*(*L*, *subs*(*x* = *x* + 1, *L3*));

```
PairsAB: number of combinations left after comparing with local data
c,s,d 3
deltaHS: Computed 1 ABPairs 4.069
```

$$G := SolOf\left(\tau - \frac{(x+1)(2x+3)}{(2x+5)(x+2)}\right)\left((x+2)^2\tau^2 - (2x+3)(17x+26)\tau + (x+2)(x+1)\right) \quad \textbf{(19)}$$

> *op*(1, *G*)

$$SolOf\left(\tau - \frac{(x+1)(2x+3)}{(2x+5)(x+2)}\right) \quad \textbf{(20)}$$

> *op*(%);

$$\tau - \frac{(x+1)(2x+3)}{(2x+5)(x+2)} \quad \textbf{(21)}$$

> *LREtools*[*OperatorToRecurrence*](%, *u*(*n*));

$$u(n+1) - \frac{(n+1)(2n+3)\,u(n)}{(2n+5)(n+2)} = 0 \quad \textbf{(22)}$$

> *LREtools*[*hypergeomsols*](%, *u*(*n*), { }, *output* = *basis*);

$$\left[\frac{1}{(n+1)(2n+3)}\right] \quad \textbf{(23)}$$

> $\frac{G}{op(1, G)} \cdot \%[1]$;

$$\frac{(x+2)^2\tau^2 - (2x+3)(17x+26)\tau + (x+2)(x+1)}{(n+1)(2n+3)} \quad \textbf{(24)}$$

```
> G := subs(n = x, %); collect(%, tau, factor);
```

$$G := \frac{(x+2)^2\,\tau^2 - (2x+3)\,(17x+26)\,\tau + (x+2)\,(x+1)}{(x+1)\,(2x+3)}$$

$$\frac{(x+2)^2\,\tau^2}{(x+1)\,(2x+3)} - \frac{(17x+26)\,\tau}{x+1} + \frac{x+2}{2x+3} \qquad \textbf{(25)}$$

```
> #A001850 :
  sq := [1, 3, 13, 63, 321, 1683, 8989, 48639, 265729, 1462563,
   8097453, 45046719, 251595969, 1409933619, 7923848253,
   44642381823, 252055236609, 1425834724419,
   8079317057869, 45849429914943, 260543813797441,
   1482376214227923, 8443414161166173,
   48141245001931263]
```

sq := [1, 3, 13, 63, 321, 1683, 8989, 48639, 265729, 1462563, 8097453, 45046719, 251595969, 1409933619, 7923848253, 44642381823, 252055236609, 1425834724419, 8079317057869, 45849429914943, 260543813797441, 1482376214227923, 8443414161166173, 48141245001931263] **(26)**

```
> seq(add(eval(coeff(G, tau, j), x = i) * sq[1 + i + j]^2, j = 0 ..2), i = 0 ..10);
```

−8, −56, −776, −14328, −305928, −7136312, −176741384, −4571103224, −122169990152, −3350375296056, −93806501688072 **(27)**

```
> #This is -8·A178808, proving that 8·A178808 has integer entries
```