```
> # Input: x = [x1, x2, x3]
  # q is an irreducible homogeneous of degree 2 in x1, x2, x3
  # p = [p1, p2, p3] is a point on q.
  #
  # The output is a list X such that the substitution {x[1]=X[1], x[2]=X[2], x[3]=X[3]}
  # turns q to the form x2^2 - x1*x3
  #
  Simplify_q := proc(q, x :: list, p :: list, X :: list) option trace;
      local x1, x2, x3, Q, i, pts, Q1, Q2, Q3, Line, Lines, EQ, v, SL, S;
      if nops(x) ≠ 3 or nops(p) ≠ 3 or p = [0, 0, 0] then
          error "wrong input"
      elif nargs = 3 then
          return procname(args, x)
      fi;
      x1, x2, x3 := op(x);
      if p[3] = 0 then
          return procname(q, x, [p[3], p[1], p[2]], subs({x1 = x2, x2 = x3, x3 = x1}, X))
      elif p[3] ≠ 1 then
          return procname(q, x, normal([p[1]/p[3], p[2]/p[3], 1]), X)
      elif p[1] ≠ 0 or p[2] ≠ 0 then
          return procname(q, x, [0, 0, 1], subs(x1 = x1 + p[1]*x3, x2 = x2 + p[2]*x3, X))
      fi;
      Q := collect(subs({x1 = X[1], x2 = X[2], x3 = X[3]}, q), {x1, x2, x3}, normal);
      if normal({seq(coeff(Q, x[i], 2), i = 1..3)}) ≠ {0} then
          pts := {[0, 0, 1]};
          # Now Q has the point [0,0,1] on it.  Lets intersect Q with three lines through
          # that point to find more points, until we have three points.
          Q1 := normal(subs(x1 = 0, Q)/x2);
          if coeff(Q1, x3) ≠ 0 then
              pts := pts union {[0, 1, normal(-coeff(Q1, x2)/coeff(Q1, x3))]}
          fi;
          Q2 := normal(subs(x2 = 0, Q)/x1);
          if coeff(Q2, x3) ≠ 0 then
              pts := pts union {[1, 0, normal(-coeff(Q2, x1)/coeff(Q2, x3))]}
          fi;
          Q3 := normal(subs(x1 = x2, Q)/x2);
          if nops(pts) < 3 and coeff(Q3, x3) ≠ 0 then
              pts := pts union {[1, 1, normal(-coeff(Q3, x2)/coeff(Q3, x3))]}
          fi;
          if nops(pts) < 3 then
              error "Input must be irreducible, not a product of two lines"
          fi;
          Line := a1 * x1 + a2 * x2 + a3 * x3;
          Lines := { };
          for i in pts do
              EQ := {seq(eval(Line, {x1 = v[1], x2 = v[2], x3 = v[3]}), v = pts minus {i})};
              Lines := Lines union {primpart(subs(solve(EQ, {a1, a2, a3}), Line), {x1, x2,
      x3})}}
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                od;
                SL := solve({seq(y[i] = Lines[i], i = 1..3)}, {x1, x2, x3});
                return procname(q, x, p, subs(seq(y[i] = x[i], i = 1..3), subs(SL, X)))
        fi;
        # Now [1,0,0], [0,1,0], [0,0,1] are on the curve.
        # So we're looking at .. x1x2 + ..x1x3 + ..x2x3 and all of those coefficients
        # are non-zero (otherwise the equation would be reducible).
        # Aiming for x2^2 - x1 * x3
        S := subs(x1 = x1 - x2 * coeff(coeff(Q, x2), x3) / coeff(coeff(Q, x1), x3), X);
        Q := collect(subs({x1 = S[1], x2 = S[2], x3 = S[3]}, q), {x1, x2, x3}, normal);
        S := subs(x3 = -x3 * lcoeff(Q, x2) / coeff(coeff(Q, x3), x1), S);
        Q := collect(subs({x1 = S[1], x2 = S[2], x3 = S[3]}, q), {x1, x2, x3}, normal);
        S := subs(x3 = x3 + x2 * coeff(coeff(Q, x2), x1) / lcoeff(Q, x2), S);
        Q := primpart(collect(subs({x1 = S[1], x2 = S[2], x3 = S[3]}, q), {x1, x2, x3}, normal),
        x2);
        if not member(x2^2 - x1 * x3, {Q, -Q}) then error "unexpected" fi;
        S;
    end:
```

> # Example:
> q := 3 * x1^2 + x1 * x2 + x1 * x3 + x2^2 - 10 * x2 * x3 - 20 * x3^2;
  Simplify_q(q, [x1, x2, x3], [3, 2, 1]);

  primpart(subs({x1 = %[1], x2 = %[2], x3 = %[3]}, q), x2);

$$q := 3\,x1^2 + x1\,x2 + x1\,x3 + x2^2 - 10\,x2\,x3 - 20\,x3^2$$

```
{--> enter Simplify_q, args = 3*x1^2+x1*x2+x1*x3+x2^2-10*x2*x3-20*
x3^2, [x1, x2, x3], [3, 2, 1]
{--> enter Simplify_q, args = 3*x1^2+x1*x2+x1*x3+x2^2-10*x2*x3-20*
x3^2, [x1, x2, x3], [3, 2, 1], [x1, x2, x3]
```
$$x1, x2, x3 := x1, x2, x3$$
```
{--> enter Simplify_q, args = 3*x1^2+x1*x2+x1*x3+x2^2-10*x2*x3-20*
x3^2, [x1, x2, x3], [0, 0, 1], [x1+3*x3, x2+2*x3, x3]
```
$$x1, x2, x3 := x1, x2, x3$$

$$Q := 3\,x1^2 + (21\,x3 + x2)\,x1 + x2^2 - 3\,x2\,x3$$

$$pts := \{[0, 0, 1]\}$$

$$Q1 := x2 - 3\,x3$$

$$pts := \left\{[0, 0, 1], \left[0, 1, \frac{1}{3}\right]\right\}$$

$$Q2 := 3\,x1 + 21\,x3$$

$$pts := \left\{[0, 0, 1], \left[0, 1, \frac{1}{3}\right], \left[1, 0, -\frac{1}{7}\right]\right\}$$

$$Q3 := 5\,x2 + 18\,x3$$

$$Line := a1\,x1 + a2\,x2 + a3\,x3$$

$$Lines := \varnothing$$

$$i := [0, 0, 1]$$

$$EQ := \left\{ a1 - \frac{a3}{7}, a2 + \frac{a3}{3} \right\}$$

$$Lines := \{3\,x1 - 7\,x2 + 21\,x3\}$$

$$i := \left[0, 1, \frac{1}{3} \right]$$

$$EQ := \left\{ a3, a1 - \frac{a3}{7} \right\}$$

$$Lines := \{x2, 3\,x1 - 7\,x2 + 21\,x3\}$$

$$i := \left[1, 0, -\frac{1}{7} \right]$$

$$EQ := \left\{ a3, a2 + \frac{a3}{3} \right\}$$

$$Lines := \{x1, x2, 3\,x1 - 7\,x2 + 21\,x3\}$$

$$SL := \left\{ x1 = y_1, x2 = y_2, x3 = \frac{y_3}{21} - \frac{y_1}{7} + \frac{y_2}{3} \right\}$$

```
{--> enter Simplify_q, args = 3*x1^2+x1*x2+x1*x3+x2^2-10*x2*x3-20*
x3^2, [x1, x2, x3], [0, 0, 1], [(4/7)*x1+(1/7)*x3+x2, (5/3)*x2+(2/21)
*x3-(2/7)*x1, (1/21)*x3-(1/7)*x1+(1/3)*x2]
```

$$x1, x2, x3 := x1, x2, x3$$

$$Q := \left(x3 + \frac{59\,x2}{7} \right) x1 - \frac{x2\,x3}{7}$$

$$S := \left[\frac{4\,x1}{7} + \frac{53\,x2}{49} + \frac{x3}{7}, \frac{239\,x2}{147} + \frac{2\,x3}{21} - \frac{2\,x1}{7}, \frac{x3}{21} - \frac{x1}{7} + \frac{46\,x2}{147} \right]$$

$$Q := \left(x3 + \frac{59\,x2}{7} \right) x1 + \frac{59\,x2^2}{49}$$

$$S := \left[\frac{4\,x1}{7} + \frac{53\,x2}{49} - \frac{59\,x3}{343}, \frac{239\,x2}{147} - \frac{118\,x3}{1029} - \frac{2\,x1}{7}, -\frac{59\,x3}{1029} - \frac{x1}{7} + \frac{46\,x2}{147} \right]$$

$$Q := \left(\frac{59\,x2}{7} - \frac{59\,x3}{49} \right) x1 + \frac{59\,x2^2}{49}$$

$$S := \left[\frac{4\,x1}{7} - \frac{6\,x2}{49} - \frac{59\,x3}{343}, \frac{121\,x2}{147} - \frac{118\,x3}{1029} - \frac{2\,x1}{7}, -\frac{59\,x3}{1029} - \frac{13\,x2}{147} - \frac{x1}{7} \right]$$

$$Q := -x1\,x3 + x2^2$$

$$\left[\frac{4\,x1}{7} - \frac{6\,x2}{49} - \frac{59\,x3}{343}, \frac{121\,x2}{147} - \frac{118\,x3}{1029} - \frac{2\,x1}{7}, -\frac{59\,x3}{1029} - \frac{13\,x2}{147} - \frac{x1}{7} \right]$$

```
<-- exit Simplify_q (now in Simplify_q) = [(4/7)*x1-(6/49)*x2-
(59/343)*x3, (121/147)*x2-(118/1029)*x3-(2/7)*x1, -(59/1029)*x3-
(13/147)*x2-(1/7)*x1]}
<-- exit Simplify_q (now in Simplify_q) = [(4/7)*x1-(6/49)*x2-
(59/343)*x3, (121/147)*x2-(118/1029)*x3-(2/7)*x1, -(59/1029)*x3-
```

```
(13/147)*x2-(1/7)*x1]}
<-- exit Simplify_q (now in Simplify_q) = [(4/7)*x1-(6/49)*x2-
(59/343)*x3, (121/147)*x2-(118/1029)*x3-(2/7)*x1, -(59/1029)*x3-
(13/147)*x2-(1/7)*x1]}
<-- exit Simplify_q (now at top level) = [(4/7)*x1-(6/49)*x2-
(59/343)*x3, (121/147)*x2-(118/1029)*x3-(2/7)*x1, -(59/1029)*x3-
(13/147)*x2-(1/7)*x1]}
```

$$\left[\frac{4\,x1}{7} - \frac{6\,x2}{49} - \frac{59\,x3}{343},\ \frac{121\,x2}{147} - \frac{118\,x3}{1029} - \frac{2\,x1}{7},\ -\frac{59\,x3}{1029} - \frac{13\,x2}{147} - \frac{x1}{7} \right]$$

$$-x1\,x3 + x2^2 \qquad\qquad\qquad\qquad \textbf{(1)}$$

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>
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