

```

> read "/Users/heba/Desktop/ORDER4/Order4-Imp/RightFactors.txt" :
> read "/Users/heba/Desktop/ORDER4/Order4-Imp/ReduceOrder.txt" :
> read "/Users/heba/Desktop/ORDER4/Order4-Imp/Hom.txt" :

>
>
> # Examples from OEIS whose recurrence has order 4 (Included in the paper "Algorithms For 2-
  Solvable Difference Equations")
> #OEIS: A227845.
> L4 := (x + 4)^2 · τ^4 - 2 · (3 · x^2 + 21 · x + 37) · τ^3 + 2 · (3 · x^2 + 15 · x + 19) · τ - (x + 2)^2;
  L4 := (x + 4)^2 τ^4 - 2 (3 x^2 + 21 x + 37) τ^3 + 2 (3 x^2 + 15 x + 19) τ - (x + 2)^2 (1)
> ReduceOrder(L4);
"Factorization..."
"Checking Symmetric Product..."
"Checking operator for u(2*n)..."

["Operator for u(2*n) is projectively equivalent to SymProd of", {τ^2 + (12 x^2 + 6 x + 1) τ (2)
  + x^2 (2 x - 1)^2, τ^2 + (12 x^2 + 18 x + 7) τ + x^2 (2 x + 1)^2}]

>
> #OEIS: A247365.
> L4 := (16 x^6 + 96 x^5 + 237 x^4 + 307 x^3 + 222 x^2 + 88 x + 15) τ^4 + (x + 1) (64 x^8 + 800 x^7
  + 4244 x^6 + 12430 x^5 + 21920 x^4 + 23837 x^3 + 15726 x^2 + 5872 x + 972) τ^3 + (
  -256 x^10 - 3840 x^9 - 25472 x^8 - 98304 x^7 - 244271 x^6 - 408233 x^5 - 464965 x^4
  - 357285 x^3 - 178432 x^2 - 53022 x - 7290) τ^2 + (-64 x^9 - 800 x^8 - 4244 x^7
  - 12422 x^6 - 21868 x^5 - 23719 x^4 - 15610 x^3 - 5847 x^2 - 1010 x - 6) τ + 16 x^6
  + 192 x^5 + 957 x^4 + 2535 x^3 + 3765 x^2 + 2977 x + 981;
L4 := (16 x^6 + 96 x^5 + 237 x^4 + 307 x^3 + 222 x^2 + 88 x + 15) τ^4 + (x + 1) (64 x^8 + 800 x^7 (3)
  + 4244 x^6 + 12430 x^5 + 21920 x^4 + 23837 x^3 + 15726 x^2 + 5872 x + 972) τ^3 + (-256 x^10
  - 3840 x^9 - 25472 x^8 - 98304 x^7 - 244271 x^6 - 408233 x^5 - 464965 x^4 - 357285 x^3
  - 178432 x^2 - 53022 x - 7290) τ^2 + (-64 x^9 - 800 x^8 - 4244 x^7 - 12422 x^6 - 21868 x^5
  - 23719 x^4 - 15610 x^3 - 5847 x^2 - 1010 x - 6) τ + 16 x^6 + 192 x^5 + 957 x^4 + 2535 x^3
  + 3765 x^2 + 2977 x + 981
> ReduceOrder(L4);
"Factorization..."
"Checking Symmetric Product..."

["Input is projectively equivalent to SymProd of", {(2 x - 1) τ^2 + 2 x (4 x^2 + 1) τ + 2 x + 1, τ^2 (4)
  + x τ - 1}]

```

> *SymProd*(*op*(%[-1]));

$$x (64x^4 + 128x^3 + 100x^2 + 36x + 17) \tau^4 - 2x(x+2) (128x^6 + 832x^5 + 2056x^4 + 2412x^3 + 1474x^2 + 591x + 174) \tau^3 - (x+1) (1024x^8 + 8192x^7 + 26880x^6 + 46592x^5 + 46140x^4 + 26864x^3 + 10013x^2 + 3162x + 690) \tau^2 + 2x(x+2) (128x^6 + 704x^5 + 1416x^4 + 1236x^3 + 506x^2 + 169x + 56) \tau + (x+2) (64x^4 + 384x^3 + 868x^2 + 876x + 345) \quad (5)$$

> *G* := *ProjectiveHom*(%, *L4*);

*G* := [ "SymProd with", 1 +  $\tau$ , " and then gauge transformation",

$$\left[ \begin{array}{l} \frac{(2x+1)\tau^3}{2(x+1)(64x^4 + 128x^3 + 100x^2 + 36x + 17)} \\ + \frac{(4x^4 + 12x^3 + 17x^2 + 13x + 5)\tau^2}{x(64x^4 + 128x^3 + 100x^2 + 36x + 17)} \\ + \frac{(2x^3 + 5x^2 + 5x + 3)\tau}{2(64x^5 + 192x^4 + 228x^3 + 136x^2 + 53x + 17)} \\ - \frac{32x^4 + 80x^3 + 82x^2 + 39x + 10}{2x(64x^4 + 128x^3 + 100x^2 + 36x + 17)} \end{array} \right]$$

>

> #OEIS: A219670

> *L4* :=  $(x+3)^2(x+4)^2(x+5)(2x+3)(7x^4 + 56x^3 + 166x^2 + 216x + 105)\tau^4 - (x+3)(x+4)(2x+3)(2x+7)(70x^6 + 1050x^5 + 6406x^4 + 20337x^3 + 35449x^2 + 32244x + 12048)\tau^3 - 3(x+3)(2x+5)(490x^8 + 9800x^7 + 84910x^6 + 416150x^5 + 1261159x^4 + 2417840x^3 + 2860095x^2 + 1905600x + 546588)\tau^2 + 27(x+2)^2(2x+3)(2x+7)(70x^6 + 1050x^5 + 6406x^4 + 20283x^3 + 35044x^2 + 31221x + 11178)\tau + 729(x+1)^3(x+2)^2(2x+7)(7x^4 + 84x^3 + 376x^2 + 744x + 550);$

*L4* :=  $(x+3)^2(x+4)^2(x+5)(2x+3)(7x^4 + 56x^3 + 166x^2 + 216x + 105)\tau^4 - (x+3)(x+4)(2x+3)(2x+7)(70x^6 + 1050x^5 + 6406x^4 + 20337x^3 + 35449x^2 + 32244x + 12048)\tau^3 - 3(x+3)(2x+5)(490x^8 + 9800x^7 + 84910x^6 + 416150x^5 + 1261159x^4 + 2417840x^3 + 2860095x^2 + 1905600x + 546588)\tau^2 + 27(x+2)^2(2x+3)(2x+7)(70x^6 + 1050x^5 + 6406x^4 + 20283x^3 + 35044x^2 + 31221x + 11178)\tau + 729(x+1)^3(x+2)^2(2x+7)(7x^4 + 84x^3 + 376x^2 + 744x + 550) \quad (7)$

> *ReduceOrder*(*L4*);

"Factorization..."

"Checking Symmetric Product..."

"Checking operator for u(2\*n)..."

"Checking Symmetric Cube... (can be time consuming...)"

["input is projectively equivalent to the symmetric cube of",  $\tau^2 + (2x + 1)\tau - 3x^2$ ] (8)

> #Example from Absolute Factorization Section in "Algorithms for 2-Solvable Difference Equations"

> L4 := (4\*x-11)\*(524160\*x^8 + 9391200\*x^7 - 118179432\*x^6 - 253541284\*x^5 - 339259113\*x^4 - 283416626\*x^3 - 140532705\*x^2 - 35130024\*x - 2220048)\*(x+5)^2\*(x+4)^2  
\*(2\*x+9)^2\*(2\*x+7)^2\*tau^4  
+ 16\*(524160\*x^12 + 15113280\*x^11 - 364158816\*x^10 - 4278491572\*x^9 - 9186978746\*x^8 + 12166953346\*x^7 - 86741410290\*x^6 - 843333775440\*x^5 - 2144077451746\*x^4 - 3001904754612\*x^3 - 2506144851117\*x^2 - 1178353117620\*x - 242095406175)\*(x+4)^2\*(2\*x+7)^2\*tau^3  
+ (-137438945280\*x^17 - 6482503372800\*x^16 - 90355220358144\*x^15 - 154953056569984\*x^14 + 8139627355615616\*x^13 + 69179680108818000\*x^12 + 277321791062784832\*x^11 + 698868352509149328\*x^10 + 1236863662787672992\*x^9 + 1625448731323698944\*x^8 + 1626145247262854144\*x^7 + 1235819925815197696\*x^6 + 686291085150978048\*x^5 + 244593652122419200\*x^4 + 24045290042818560\*x^3 - 27607241721839616\*x^2 - 15602879836717056\*x - 2930851407200256)\*tau^2  
- 32768\*(1048320\*x^12 + 38613120\*x^11 - 475672512\*x^10 - 11499544808\*x^9 - 68147233556\*x^8 - 184773020492\*x^7 - 262836346620\*x^6 - 216526023556\*x^5 - 122659285853\*x^4 - 39783078178\*x^3 + 1029344695\*x^2 + 7429526904\*x + 2484513522)\*(x+2)^2\*(2\*x+3)^2\*tau  
+ 4096\*(4\*x+33)\*(524160\*x^8 + 13584480\*x^7 - 37764552\*x^6 - 736049716\*x^5 - 3014273813\*x^4 - 6181409598\*x^3 - 7122549901\*x^2 - 4430333096\*x - 1162363872)\*(2\*x+3)^2\*(2\*x+1)^2\*(x+2)^2\*(x+1)^2;

L4 := (4x - 11) (524160 x^8 + 9391200 x^7 - 118179432 x^6 - 253541284 x^5 - 339259113 x^4 - 283416626 x^3 - 140532705 x^2 - 35130024 x - 2220048) (x + 5)^2 (x + 4)^2 (2x + 9)^2 (2x + 7)^2 tau^4 + 16 (524160 x^12 + 15113280 x^11 - 364158816 x^10 - 4278491572 x^9 - 9186978746 x^8 + 12166953346 x^7 - 86741410290 x^6 - 843333775440 x^5 - 2144077451746 x^4 - 3001904754612 x^3 - 2506144851117 x^2 - 1178353117620 x - 242095406175) (x + 4)^2 (2x + 7)^2 tau^3 + (-137438945280 x^17 - 6482503372800 x^16 - 90355220358144 x^15 - 154953056569984 x^14 + 8139627355615616 x^13 + 69179680108818000 x^12 + 277321791062784832 x^11 + 698868352509149328 x^10 + 1236863662787672992 x^9 + 1625448731323698944 x^8 + 1626145247262854144 x^7 + 1235819925815197696 x^6 + 686291085150978048 x^5 + 244593652122419200 x^4 + 24045290042818560 x^3 - 27607241721839616 x^2 - 15602879836717056 x - 2930851407200256) tau^2 - 32768 (1048320 x^12 + 38613120 x^11 - 475672512 x^10

$$\begin{aligned}
& - 11499544808 x^9 - 68147233556 x^8 - 184773020492 x^7 - 262836346620 x^6 \\
& - 216526023556 x^5 - 122659285853 x^4 - 39783078178 x^3 + 1029344695 x^2 \\
& + 7429526904 x + 2484513522) (x + 2)^2 (2x + 3)^2 \tau + 4096 (4x + 33) (524160 x^8 \\
& + 13584480 x^7 - 37764552 x^6 - 736049716 x^5 - 3014273813 x^4 - 6181409598 x^3 \\
& - 7122549901 x^2 - 4430333096 x - 1162363872) (2x + 3)^2 (2x + 1)^2 (x + 2)^2 (x + 1)^2
\end{aligned}$$

```
> t0 := time( );
                                t0 := 260.706                                (10)
```

```
> _Env_print_number_cases := true;
                                _Env_print_number_cases := true          (11)
```

```
> ReduceOrder(L4);
"Factorization..."
"Number of cases", 20
"Number of cases", 30
"Number of cases", 20
"Absolute factorization..."
"Number of cases", 121
```

```
[ "Absolute factorization, right factors of operator for u(2*n) are", { (8x - 15) (4x + 5)^2 (2x + 3)^2 (4x + 7)^2 (x + 2)^2 \tau^2 - 1024 (32776 x^2 + 331857 x + 1317869) x^7 \tau + 4096 (8x + 33) (4x + 1)^2 (2x + 1)^2 (4x + 3)^2 (x + 1)^2, (8x - 11) (2x + 5)^2 (4x + 7)^2 (x + 2)^2 (4x + 9)^2 \tau^2 - 4 (65552 x^2 + 729266 x + 2983983) (2x + 1)^7 \tau + 4096 (8x + 37) (4x + 5)^2 (2x + 3)^2 (4x + 3)^2 (x + 1)^2 } ] (12)
```

```
> lprint(time( ) - t0);
12.188
```

```
> # 24.702 seconds for 1791 cases (use neither)
> # 11.938 seconds for 597 cases (use DetFactorsSelect)
> # 6.950 seconds for 363 cases (use DeterminantSelect)
> # 3.185 seconds for 121 cases (use both)
```

```
> #Examples from OEIS whose recurrence has order 3 and 4 (Examples included in the paper
    "Solving Third Order Linear Difference Equations in Terms of Second Order Equations")
```

```
> #OEIS A260772
```

```
> L4 := (x + 5) * (x + 4) * (25 * x^2 + 130 * x + 141) * tau^4 - 30 * (x + 4) * (7 * x + 13)
    * tau^3 + (-1100 * x^4 - 12320 * x^3 - 48664 * x^2 - 80740 * x - 47400) * tau^2
    + 120 * (x + 6) * (x + 1) * tau - 16 * (x + 1) * (25 * x^2 + 180 * x + 296) * x;
```

```
L4 := (x + 5) (x + 4) (25 x^2 + 130 x + 141) \tau^4 - 30 (x + 4) (7 x + 13) \tau^3 + (-1100 x^4 (13)
```

$$-12320x^3 - 48664x^2 - 80740x - 47400) \tau^2 + 120(x+6)(x+1)\tau - 16(x+1)(25x^2 + 180x + 296)x$$

```
> ReduceOrder(L4);
"Factorization..."
"Absolute factorization..."
"Number of cases", 3
```

$$\left[ \text{"Absolute factorization, right factors of operator for } u(2*n) \text{ are", } \left\{ (2x+5)(5x+3)(x+2)\tau^2 \right. \right. \quad (14)$$

$$\left. \left. + (-440x^3 - 1584x^2 - 1780x - 600)\tau - 16(5x+8)(2x+1)x, (2x+5)(10x+9)(x+2)\tau^2 + (-880x^3 - 3432x^2 - 4220x - 1650)\tau - 16(10x+19)(2x+1)x \right\} \right]$$

```
> # OEIS A295371
```

```
> L3 := (2*x + 3) * (x + 4)^2 * tau^3 - (2*x + 3) * (7*x^2 + 52*x + 97) * tau^2 - 3 * (2*x + 7) * (7*x^2 + 18*x + 12) * tau + 27 * (2*x + 7) * (x + 1)^2;
```

$$L3 := (2x+3)(x+4)^2\tau^3 - (2x+3)(7x^2+52x+97)\tau^2 - 3(2x+7)(7x^2+18x+12)\tau + 27(2x+7)(x+1)^2 \quad (15)$$

```
> ReduceOrder(L3, `projective`);
"Factorization..."
"Number of cases", 3
"Number of cases", 3
"Calling ReduceOrder32..."
"Number of cases", 5
```

$$\tau^2 + (2x+1)\tau - 3x^2 \quad (16)$$

```
> # Solve it in terms of OEIS entry A002426: (x+2)*tau^2 + (-2*x-3)*tau-3*x-3;
```

```
> SimpIB((x+2)*tau^2 + (-2*x-3)*tau-3*x-3, `projective`);
```

$$\tau^2 + (-2x-1)\tau - 3x^2 \quad (17)$$

```
> G := ProjectiveHom(sympower((x+2)*tau^2 + (-2*x-3)*tau-3*x-3, 2), L3);
```

$$G := \left[ \text{"SymProd with", } \tau - 1, \text{"} \right] \text{ and then gauge transformation, } -\frac{81\tau}{10} - \frac{243}{10} \quad (18)$$

```
>
```

```
> # OEIS A178808
```

```
> L3 := (x + 2) * (2*x + 1) * (x + 3)^2 * tau^3 - (x + 2) * (2*x + 1) * (35*x^2 + 141*x + 134) * tau^2 + (2*x + 5) * (x + 1) * (35*x^2 + 69*x + 26) * tau - (2*x + 5) * (x + 1) * x^2;
```

$$L3 := (x+2)(2x+1)(x+3)^2\tau^3 - (x+2)(2x+1)(35x^2+141x+134)\tau^2 + (2x+5)(x+1)(35x^2+69x+26)\tau - (2x+5)(x+1)x^2 \quad (19)$$

```
> ReduceOrder(L3, `projective`);
"Factorization..."
"Number of cases", 1
"Number of cases", 1
```

```
"Calling ReduceOrder32..."
```

```
"Number of cases", 1
```

$$\tau^2 + (6x + 3)\tau + x^2 \quad (20)$$

```
> # OEIS A268138
```

```
> L3 := (x + 4) * (2*x + 3) * (x + 3)^2 * tau^3 - (2*x + 5) * (x + 3) * (35*x^2 + 107*x + 82) * tau^2 + (2*x + 3) * (x + 1) * (35*x^2 + 173*x + 214) * tau - (2*x + 5) * (x + 1)^2 * x;
```

$$L3 := (x + 4) (2x + 3) (x + 3)^2 \tau^3 - (2x + 5) (x + 3) (35x^2 + 107x + 82) \tau^2 + (2x + 3) (x + 1) (35x^2 + 173x + 214) \tau - x (x + 1)^2 (2x + 5) \quad (21)$$

```
> ReduceOrder(L3, `projective`);
```

```
"Factorization..."
```

```
"Number of cases", 1
```

```
"Number of cases", 1
```

```
"Calling ReduceOrder32..."
```

```
"Number of cases", 1
```

$$\tau^2 + (6x + 3)\tau + x^2 \quad (22)$$

```
>
```

```
>
```

```
>
```

```
>
```

```
>
```

*#Example whose recurrence has order 4 (The example is from the file Absolute\_Factorization.mws in [https://www.math.fsu.edu/~hboukaed/Implementations/Worksheets%20for%20ISSAC%20paper%20\(Order3\)/](https://www.math.fsu.edu/~hboukaed/Implementations/Worksheets%20for%20ISSAC%20paper%20(Order3)/))*

```
> L4 := (16*x + 32) * tau^4 + (-16*x - 40) * tau^3 + (4*x + 12) * tau^2 - 2 * (2*x + 3) * (x + 3) * (x + 2) * tau - (x + 3) * (x + 1) * (x + 2)^2 * x;
```

$$L4 := (16x + 32) \tau^4 + (-16x - 40) \tau^3 + (4x + 12) \tau^2 - 2 (2x + 3) (x + 3) (x + 2) \tau - (x + 3) (x + 1) (x + 2)^2 x \quad (23)$$

```
> ReduceOrder(L4);
```

```
"Factorization..."
```

```
"Number of cases", 1
```

```
"Number of cases", 2
```

```
"Number of cases", 1
```

```
"Absolute factorization..."
```

```
"Number of cases", 3
```

```
[ "Absolute factorization, right factors of operator for u(2*n) are", { \tau^2 + (-x - 2) \tau - (x + 1)^2 (x + 2) x, 16 \tau^2 + (-16x - 24) \tau - (2x + 3)^2 (2x + 1)^2 } ] \quad (24)
```