Solving Third Order Linear Difference Equations in Terms of Second Order Equations

Heba Bou KaedBey, Mark van Hoeij, and Man Cheung Tsui

Florida State University

Talk presented by Heba Bou KaedBey

Recurrence Operators

Definition

Let $D = \mathbb{C}(x)[\tau] = \left\{ \sum_{i=0}^{k} a_i \tau^i \,\middle|\, a_i \in \mathbb{C}(x) \right\}$. A **recurrence operator** is an element $L \in D$. Here $L = \sum_{i=0}^{k} a_i \tau^i$ acts on a function $f(x)$ as

$$L(f)(x) = \sum_{i=0}^{k} a_i f(x+i).$$

In particular, $\tau \in D$ is the **shift operator** $\tau(f)(x) = f(x+1)$.

Multiplication in the ring D is defined as composition of operators, e.g. if $a \in \mathbb{C}(x) \subseteq D$ then $\tau \cdot a = \tau(a)\tau \in D$.

Recurrence Operators

A solution of L is a function $f(x)$ with $L(f) = 0$. i.e
$a_k f(x+k) + \cdots + a_0 f(x) = 0$.
We say f is a rational solution if $f \in \mathbb{C}(x)$.

If $\frac{f(x+1)}{f(x)} \in \mathbb{C}(x)$ then $f(x)$ corresponds to a right factor $\tau - r$ of L where $r = \frac{f(x+1)}{f(x)}$.

More generally, we consider solutions in a $D-$module S defined as $S = \mathbb{C}^{\mathbb{N}} / \sim$ where two sequences $u, v \in \mathbb{C}^{\mathbb{N}}$ are equivalent if $u - v : \mathbb{N} \longrightarrow \mathbb{C}$, has finite support.

Recurrence Operators

Definition

The **solution space** $V(L)$ of operator L is the set $\left\{u \in S \,\middle|\, L(u) = 0\right\}$.

This is a $\mathbb{C}-$vector space of dim $\operatorname{ord}(L)$:

(Key Property of S)

If $L = \sum_{i=0}^{k} a_i \tau^i$ with $a_0 \neq 0$ and $a_k \neq 0$, then L has k linearly independent solutions in S.

Liouvillian operator

Definition

A difference operator is said to be **Liouvillian**, if it has a non-zero Liouvillian solution built from

- solutions of order $\mathbf{1}$ operators
- $+, \cdot$, shift
- indefinite sum (defined on next slide)
- interlacing (defined on next slide)

Definition (Indefinite Sum)

v is an **indefinite sum** of u if $(\tau - 1)v = u$.

Definition (Interlacing)

Let $A_i = (a_{i,0}, a_{i,1}, a_{i,2}, \ldots)$. The **interlacing** of $A_1, \ldots, A_k$ is the sequence $A := (a_{1,0}, \ldots, a_{k,0}, a_{1,1}, \ldots, a_{k,1}, \ldots)$.

Example (Interlacing)

The interlacing of $u = (u_0, u_1, \ldots)$ and $v = (v_0, v_1, \ldots)$ is $(u_0, v_0, u_1, v_1, \ldots)$.

Definition (Least Common Left Multiple)

Let $L_1, L_2 \in D$. **LCLM(L_1,L_2)** is the unique monic generator of $DL_1 \cap DL_2$.

Definition (Symmetric Product)

Let $L_1, L_2 \in D$. The **symmetric product** $L_1 \circledS L_2$ of L_1 and L_2 is defined as the monic operator $L \in D$ of smallest order such that $L(u_1 u_2) = 0$ for all $u_1, u_2 \in S$ with $L_1 u_1 = 0$ and $L_2 u_2 = 0$.

Definition (Symmetric Square)

$L^{\circledS 2} := L \circledS L$

Liouvillian Seq closed under

- Solns of order 1 operators
- $u_1 + u_2$
- $u_1 \cdot u_2$
- $\sum u_1$
- $\mathrm{Intl}(u_1, \ldots, u_n)$

Liouvillian Op closed under

$\tau - r_1, \tau - r_2, \ldots$

$L_1, L_2 \rightsquigarrow \mathrm{LCLM}(L_1, L_2)$

$L_1, L_2 \rightsquigarrow L_1 \circledS L_2$

$L_1 \rightsquigarrow L_1 \cdot (\tau - 1)$

$L_1, L_2 \rightsquigarrow \mathrm{Intl}(L_1, \ldots, L_n)$

Factors

2-Expressible Seq closed under

- Solns of order 2 operators
- $u_1 + u_2$
- $u_1 \cdot u_2$
- $\sum u_1$
- $\mathrm{Intl}(u_1, \ldots, u_n)$

2-Solvable Op closed under

$a_2\tau^2 + a_1\tau + a_0, \ldots$

$L_1, L_2 \rightsquigarrow \mathrm{LCLM}(L_1, L_2)$

$L_1, L_2 \rightsquigarrow L_1 \circledS L_2$

$L_1 \rightsquigarrow L_1 \cdot (\tau - 1)$

$L_1, L_2 \rightsquigarrow \mathrm{Intl}(L_1, \ldots, L_n)$

Factors

OEIS Example: A295371. IS IT 2-SOLVABLE?

The sequence is

$$a(n) = \frac{1}{2n} \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} \binom{2k}{k} (k+2)(-3)^{(n-1-k)}.$$

OEIS list the recurrence:

$$L_3 = (2x+1)(x+3)^2\tau^3 - (2x+1)(7x^2+38x+52)\tau^2 - 3(2x+5)(7x^2+4x+1)\tau + 27(2x+5)x^2$$

equivalently,

$$(2n+1)(n+3)^2 a(n+3) = (2n+1)(7n^2+38n+52)a(n+2) + 3(2n+5)(7n^2+4n+1)a(n+1) - 27(2n+5)n^2 a(n)$$

Zhi-Wei-Sun conjectured that $a(n)$ is a positive odd integer for all $n > 0$.

OEIS Example: A295371

For the order 3 recurrence L_3, we have two cases to consider:

Case 1: If L_3 is solvable, then how to find such solutions ?

We need an algorithm. That is the topic of this talk.

Case 2: If L_3 is not 2-solvable, proving that requires Galois theory:

- Difference Case: Heba Bou KaedBey, Mark van Hoeij, and Man Cheung Tsui, 2024.
- Differential Case: Michael F Singer, 1985.

Main Theorem

Theorem (Theorem 7.1 in Bou KaedBey, van Hoeij, and Tsui 2024)

Let $L \in D$ be an order 3 linear difference operator. Then, one of the following holds.

1. *(Reducible Case) L admits a non trivial factorization over $\mathbb{C}(x)$.*
2. *(Liouvillian Case) L is irreducible but has Liouvillian solutions.*
3. *L is gauge equivalent to $L_2^{ⓢ2} ⓢ (\tau - r)$ for some order 2 operator L_2 and $r \in \mathbb{C}(x)$. (will define gauge equivalent in the next slide)*
4. *Not 2-solvable.*

Gauge Transformation

Definition (Gauge Transformation)

Let $D = \mathbb{C}(x)[\tau]$. If $L \in D \setminus \{0\}$ then D/DL is a $D-$module. L_1 is **gauge equivalent** to L_2 when

- D/DL_1 and D/DL_2 are isomorphic as $D-$modules.
  or equivalently,
- there exists $G \in D$ such that $G(V(L_2)) = V(L_1)$ and L_1, L_2 have the same order.

This G defines a bijection $V(L_2) \to V(L_1)$. This bijection is called a **gauge transformation**.

Goal: Reduce order 3 recurrence L_3 to an order 2 recurrence L_2.

$$L_3 \rightsquigarrow L_2^{Ⓢ2} Ⓢ (\tau - r)$$

when possible (Case 3).

Motivation: Solvers for order 2:

Yongjae Cha, Mark Van Hoeij, and Giles Levy. "Solving recurrence relations using local invariants." Proceedings of ISSAC'10.

ReduceOrder Algorithm

We first compute $L_3^{\circledS 2}$, which will have order 5 or 6:

Case 1 (Simplest Case): $\text{ord}(L_3^{\circledS 2}) = 5$

Theorem

Let $L_3 = \tau^3 + c_2\tau^2 + c_1\tau^1 + c_0$ be a difference operator over $\mathbb{C}(x)$. $L_3 = (\tau^2 + \tau + b)^{\circledS 2} \circledS (\tau - r)$ for some $b \in \mathbb{C}(x) \setminus \{0, 1\}$ and $r \in \mathbb{C}(x) \setminus \{0\} \iff L_3^{\circledS 2}$ has order 5.

Theorem 5.1 in our ISSAC paper directly gives us a formula for b and r. That formula effectively covers this case.

Case 2 : $\mathrm{ord}(L_3^{Ⓢ2}) = 6$

Goal: To reduce:

$$L_3 \rightsquigarrow L_{\text{want}}$$

$$\text{Case 2} \rightsquigarrow \text{want Case 1}$$

$$\mathrm{ord}(L_3^{Ⓢ2}) = 6 \rightsquigarrow \mathrm{ord}(L_{\text{want}}^{Ⓢ2}) = 5$$

solved by Theorem 5.1

Let $L_3 = a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0$.

$G = b_0 + b_1\tau + b_2\tau^2$, where the b_i's are unknowns in $\mathbb{C}(x)$.

Commutative Diagram (If in Case 3)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V(L_1) & \longrightarrow & V(L_3^{ⓢ2}) & \xrightarrow{G_2} & V(L_{want}^{ⓢ2}) & \longrightarrow & 0 \\
 & & & & \uparrow {\scriptstyle u \mapsto u^2} & & \uparrow {\scriptstyle v \mapsto v^2} & & \\
 & & & & V(L_3) & \xrightarrow{G} & V(L_{want}) & &
\end{array}$$

Commutative Diagram (If in Case 3)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V(L_1) & \longrightarrow & V(L_3^{ⓢ2}) & \xrightarrow{G_2} & V(L_{want}^{ⓢ2}) & \longrightarrow & 0 \\
 & & & & {\scriptstyle u^2 \atop {\uparrow \atop u}} \Big\uparrow & & {\scriptstyle v^2 \atop {\uparrow \atop v}} \Big\uparrow & & \\
 & & & & V(L_3) & \xrightarrow{G} & V(L_{want}) & &
\end{array}
$$

The induced map G_2 is a map from a 6 dimensional vector space $V(L_3^{ⓢ2})$ to a 5 dimensional vector space $V(L_{\text{want}}^{ⓢ2})$.

Commutative Diagram (If in Case 3)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V(L_1) & \longrightarrow & V(L_3^{\circledS 2}) & \xrightarrow{G_2} & V(L_{want}^{\circledS 2}) & \longrightarrow & 0 \\
 & & & & {\scriptstyle u^2 \uparrow u}\Big\uparrow & & {\scriptstyle v^2 \uparrow v}\Big\uparrow & & \\
 & & & & V({\color{blue}L_3}) & \xrightarrow{\color{red}G} & V({\color{red}L_{want}}) & &
\end{array}
$$

This means that G_2 should have a kernel. Since the map G_2 is onto, and we are going from a 6 dimensional vector space to a 5 dimensional vector space, then the kernel, $V(L_1)$, should have dimension 1.

This means that if in case 3, $L_3^{\circledS 2}$ has an order 1 right factor L_1.

If there is no right factor of order 1, then we are not in case 3, and the algorithm stops.

$G_2(\text{sol of} L_1) = 0 \rightsquigarrow$ quadratic equation in b_0, b_1, b_2.

$\downarrow$

Using "Conic Solver"

$\downarrow$

$b_0, b_1, b_2 \in \mathbb{C}(x)$

$\downarrow$

now, $G = b_0 + b_1\tau + b_2\tau^2$ is known

$\downarrow$

get L_{want}

$\downarrow$ by theorem 5.1

find L_2 and r

OEIS A295371 (order 3 irreducible)

$\downarrow$ our program

L_2 (order 2)

$\downarrow$

A formula for $A295371(n)$ in terms of a solution $u \in V(L_2)$

$\downarrow$

Proves $A295371(n) \in \mathbb{Z}$? Is $u(n) \in \mathbb{Z}$?

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Proves $A295371(n) \in \mathbb{Z}$? Is $u(n) \in \mathbb{Z}$?

Idea: $L_2 \rightsquigarrow$ something in the OEIS.

OEIS A295371 (order 3 irreducible)

↓ our program

L_2 (order 2)

↓ Giles implementation

Solves L_2 in terms of *A001006* , *A002426*, and many more

This way A295371 is written in terms of known integer sequences.

OEIS A295371 (order 3 irreducible)

↓ our program

L_2 (order 2)

↓ Giles implementation

Solves L_2 in terms of *A001006*, A002426, and many more

This way A295371 is written in terms of known integer sequences.

Tried several: A002426 gives the smallest formula, namely $A295371(n) = \frac{b(n)^2+3b(n-1)^2}{4}$, where $b(n)$ is A002426.

1. Background and Definitions
2. Example
3. Main Theorem
4. Summary and Closure

$$\begin{array}{c}\text{differential equation or recurrence relation}\\ \downarrow \\ \text{solvers} \\ \downarrow \\ \text{closed form expressions or other formulas} \\ \downarrow \\ \text{proofs}\end{array}$$

- There are a number of solvers for differential equations and recurrence relations of order 2. That motivates the question of when can we reduce order 3 to order 2 (Goal in paper).
- Next Goal: Reduce order 4 to order 2:
  - Absolute Factorization (in paper)
  - $\cdots$ (work in progress)(differential Galois theory results $\rightsquigarrow$ difference case)

Thank You!

References I

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