Study Questions Homework 3 Foundations of Computational Math 2 Spring 2025

Advanced Question on Splines

Problem 3.1

Consider a set of equidistant mesh points, $x_k = x_0 + kh$, $0 \leq k \leq m$.

3.1.a. Determine a cubic spline $b_i(x)$ that satisfies the following conditions:

$$b_i(x_j) = \begin{cases} 0 & \text{if } j < i-1 \text{ or } j > i+1 \\ 1 & \text{if } j = i \end{cases}$$

$$b_i'(x) = b_i''(x) = 0 \quad \text{for } x = x_{i-2} \text{ and } x = x_{i+2}$$

(For simplicity, you may assume that $2 < i < m-2$.)

3.1.b. Show that $b_i(x) = 0$ when $|x - x_i| \geq 2h$.

3.1.c. Show that $b_i(x) > 0$ when $|x - x_i| < 2h$.

3.1.d. What is the relationship between the spline, $b_i(x)$, and a B-spline?

Intermediate Questions on Approximation and Economization

Problem 3.2

Consider $f(x) = e^x$ on the interval $-1 \leq x \leq 1$. Suppose we want to approximate $f(x)$ with a polynomial. Generate the following polynomials:

(a) $F_1(x)$ and $F_3(x)$:the first and third order Taylor series approximations of $f(x)$ expanded about $x = 0$.

(b) $N_1(x)$: the linear near-minimax approximation to $f(x)$ on the interval.

(c) $C_1(x)$ and $C_2(x)$ – the linear and quadratic polynomials that result from Chebyshev economization applied to $F_3(x)$, the third order Taylor series approximation of $f(x)$ expanded about $x = 0$.

(d) $p_1(x)$ and $p_2(x)$ – the linear and quadratic polynomials that result from Legendre economization applied to $F_3(x)$, the third order Taylor series approximation of $f(x)$ expanded about $x = 0$.

(3.2.a) Derive bounds on the ∞ norm of the error where possible.

(3.2.b) Evaluate the error for each polynomial approximation on a very fine grid on the interval $-1 \leq x \leq 1$ and compare to the bounds.

Problem 3.3

This is a study question not a programming assignment and you need not turn in any code. This problem considers the use of discrete least squares for approximation by a polynomial. Recall, the distinct points $x_0 < x_1 < \cdots < x_m$ are given and the **discrete** metric

$$c(p_n) = \sum_{i=0}^{m} \omega_i (f(x_i) - p_n(x_i))^2$$

with $\omega_i > 0$ is used to determine the polynomial, $p_n^{ls}(x)$, of degree n that achieves the minimal value.

Assume that $\omega_i = 1$ for this exercise.

This means that

$$c(p_n) = \sum_{i=0}^{m} \omega_i (f(x_i) - p_n(x_i))^2 = \|F - P_n\|_2^2,$$

where, $p_n(x) \in \mathbb{P}_n$,

$$F = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \end{pmatrix} \in \mathbb{R}^{m+1}, \quad \text{and} \quad P_n = \begin{pmatrix} p_n(x_0) \\ p_n(x_1) \\ \vdots \\ p_n(x_m) \end{pmatrix} \in \mathbb{R}^{m+1}$$

and the norm is the standard Euclidean norm, i.e., the 2-norm, $\forall v \in \mathbb{R}^{m+1}, \ \|v\|_2^2 = v^T v$.

Typically, $m \gg n$. If $m = n$ then the unique interpolating polynomial is the solution.

Since the optimization problem is over all $p_n \in \mathbb{P}_n$, if we parameterize $\mathbb{P}_n$ as

$$p_n(x) = \sum_{j=0}^{n} \phi_j(x)\gamma_j$$

then the conditions are

$$\begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_m \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \end{pmatrix} - \begin{pmatrix} \phi_0(x_0) & \dots & \phi_n(x_0) \\ \phi_0(x_1) & \dots & \phi_n(x_1) \\ \vdots & & \vdots \\ \phi_0(x_m) & \dots & \phi_n(x_m) \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$$r = b - Ag$$

and the optimization problem becomes

$$\min_{g \in \mathbb{R}^{n+1}} \|b - Ag\|_2^2,$$

i.e., minimize the residual r as a function of $g \in \mathbb{R}^{n+1}$.

Use the Chebyshev polynomials to form an orthonormal basis, i.e.,

$$\phi_i(x) = \alpha_i T_i(x)$$

and the roots of $T_{m+1}(x)$ as the x_i.

1. Identify the important property that the matrix possesses that allows the system to be be solved in $O(n)$ comptuations.

2. Verify empirically that the matrix satisfies the property above to numerical precision.

3. Use your solution to implement a code that assembles the least squares problem and solves it to find the optimal solution $g_* \in \mathbb{R}^{n+1}$ Make sure to exploit the algebraic properties of the matrix A to have an efficient solution.

4. Apply your code to several $f(x)$ choices and use multiple n and m values to explore the accuracy of the approximation. Approximate $\|f - p_n^{ls}\|_\infty$ by sampling the difference between f and the polynomial at a large number of points in the interval and taking the maximum magnitude.

5. For each, problem you solve check the residual of the overdetermined system $r = (b - Ag_*$ where g_* is the optimal set of coefficients. Empirically evaluate how it relates to the subspace $\mathcal{R}(A)$.

Basic and Intermediate Questions on Orthogonal Polynomials

Problem 3.4

3.4.a. Suppose you are given an arbitrary polynomial of degree 3 or less with the form

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3.$$

Show that there are unique coefficients, γ_i, $0 \leq i \leq 3$, for $p(x)$ in the representation of the form

$$p(x) = \gamma_0 T_0(x) + \gamma_1 T_1(x) + \gamma_2 T_2(x) + \gamma_3 T_3(x)$$

where $T_i(x)$, $0 \leq i \leq 3$, are the Chebyshev polynomials.

3.4.b. Is this true for any degree n? Justify your answer.

3.4.c. Consider $T_{32}(x)$, the Chebyshev polynomial of degree 32 and $T_{51}(x)$, the Chebyshev polynomial of degree 51. What is the coefficient of x^{13} in $T_{32}(x)$? What is the coefficient of x^{20} in $T_{51}(x)$?

Problem 3.5

For this problem, consider the space $\mathcal{L}^2[-1,1]$ with inner product and norm

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx \text{ and } \|f\|^2 = (f,f)$$

Let $P_i(x)$, for $i = 0, 1, \ldots$ be the Legendre polynomials of degree i and let $n+1-$st have the form

$$P_{n+1}(x) = \rho_n (x - x_0)(x - x_1)\cdots(x - x_n)$$

i.e., x_i for $0 \leq i \leq n$ are the roots of $P_{n+1}(x)$.

Let the Lagrange interpolation functions that use the x_i be $\ell_i(x)$ for $0 \leq i \leq n$. So, for example,

$$L_n(x) = \ell_0(x)f(x_0) + \cdots + \ell_n(x)f(x_n)$$

is the Lagrange form of the interpolation polynomial of $f(x)$ defined by the roots.

Let $\mathbb{P}_n$ be the space of polynomials of degree less than or equal to n. We can write the least squares approximation of $f(x)$ in terms of the $P_i(x)$ using the generalized Fourier series as

$$f_n(x) = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \cdots + \alpha_n P_n(x) \text{ where } \alpha_i = \frac{(f, P_i)}{(P_i, P_i)}$$

3.5.a

Clearly, $(\ell_i, \ell_i) \neq 0$. Show that $(\ell_i, \ell_j) = 0$ when $i \neq j$. Therefore, the functions $\ell_0(x), \ldots, \ell_n(x)$ are an orthogonal basis for $\mathbb{P}_n$.

3.5.b

Suppose we evaluate $f_n(x)$ at the x_i to obtain the data $f_n(x_0), \ldots, f_n(x_n)$. We can then write $f_n(x)$ in its Lagrange form,

$$f_n(x) = L_n(x) = f_n(x_0)\ell_0(x) + \ldots + f_n(x_n)\ell_n(x)$$

Since the $\ell_0(x), \ldots, \ell_n(x)$ are an orthogonal basis for $\mathbb{P}_n$, they also can be used to compute, $f_n(x)$, the unique least squares approximation to $f(x)$. As with the Legendre polynomials, using the generalized Fourier series, yields

$$f_n(x) = \sigma_0\ell_0(x) + \sigma_1\ell_1(x) + \cdots + \sigma_n\ell_n(x) \text{ where } \sigma_i = \frac{(f, \ell_i)}{(\ell_i, \ell_i)}$$

Show that these two forms of $f_n(x)$ give the same polynomial by showing that

$$\sigma_i = \frac{(f, \ell_i)}{(\ell_i, \ell_i)} = f_n(x_i)$$

Hint: Consider the relationship between $f(x)$ and $f_n(x)$.

Problem 3.6

Suppose $\phi_i(x)$, $i = 0, 1, \ldots$ are a set of real orthonormal polynomials with respect to the inner product

$$(f, g)_\omega = \int_a^b \omega(x) f(x) g(x)\, dx$$

where $\phi_i(x)$ has degree i. Show that

$$\int_a^b \omega(\xi) G_n(x, \xi)\, d\xi = 1$$

where

$$G_n(x, \xi) = \sum_{i=0}^{n} \phi_i(x)\phi_i(\xi)$$

Important Intermediate Question on Orthogonal Polynomial Recurrences

Problem 3.7

For the Legendre polynomials, $P_n(x)$, we have the recurrence

$$P_0 = 1, \quad P_1 = x$$
$$P_{n+1} = \frac{2n+1}{n+1} x P_n - \frac{n}{n+1} P_{n-1}$$

This familiar recurrence yields a form that is orthogonal with respect to the inner product $(f, g) = \int_{-1}^{1} g(x) f(x) dx$, but it is not monic and it is not orthonormal. We have

$$(P_n, P_n) = \frac{2}{2n+1}$$

$$P_2 = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_3 = \frac{5}{2}x^3 - \frac{3}{2}x$$

3.7.a

Let $\tilde{P}_n(x)$ be the Legendre polynomial of degree n that is normalized so that the series is orthonormal, i.e.,

$$(\tilde{P}_i,\ \tilde{P}_j) = \delta_{ij}$$

but not necessarily in monic form. Derive a recurrence that relates $\tilde{P}_{n+1}$ to $\tilde{P}_n$ and $\tilde{P}_{n-1}$.

3.7.b

Recall, that the class notes and the reference text by Isaacson and Keller give the following recurrence for the normalized (but not monic) orthogonal polynomials

$$\tilde{P}_{n+1} = (A_n x + B_n)\tilde{P}_n - C_n \tilde{P}_{n-1}$$

where

$$\tilde{P}_n = \tilde{a}_n x^n + \tilde{b}_n x^{n-1} + q_{n-2}(x)$$

$$A_n = \frac{\tilde{a}_{n+1}}{\tilde{a}_n}$$

$$B_n = \frac{\tilde{a}_{n+1}}{\tilde{a}_n}\left(\frac{\tilde{b}_{n+1}}{\tilde{a}_{n+1}} - \frac{\tilde{b}_n}{\tilde{a}_n}\right)$$

$$C_n = \frac{\tilde{a}_{n+1}\tilde{a}_{n-1}}{\tilde{a}_n^2}$$

Show that your recurrence from previous part of the problem is equivalent to this recurrence. when it is applied to the Legendre polynomials.

3.7.c

The textbook of Quarteroni et al. in equations (10.7) and (10.8) gives the recurrence for the monic, but not necessarily normalized, form of Legendre polynomials. (In fact, this recurrence holds for any family of orthogonal monic polynomials and where the coefficients are determined by the inner product associated with the particular family.) The coefficients of this recurrence gives the coefficients necessary to define the so-called Jacobi matrix whose eigendecomposition give the Gauss Legendre quadrature nodes and weights. Determine the values of α_n and β_{n+1} and show that they are consistent with the standard recurrence given at the start of the problem and the values used in the MATLAB codes for Gauss Legendre quadrature in Section 10.6 of Quarteroni et al. (see the class website for a scan of those pages of their textbook).

The recurrence from Quateroni et al. due to Gautschi is

$$\hat{P}_{n+1} = (x - \alpha_n)\hat{P}_n - \beta_n \hat{P}_{n-1}, \quad n \geq 0$$

$$\alpha_n = \frac{(x\hat{P}_n, \hat{P}_n)_\omega}{(\hat{P}_n, \hat{P}_n)_\omega}$$

$$\beta_{n+1} = \frac{(\hat{P}_{n+1}, \hat{P}_{n+1})_\omega}{(\hat{P}_n, \hat{P}_n)_\omega}$$

$$\hat{P}_{-1} = 0, \quad \hat{P}_0 = 1.$$

Note that since $\hat{P}_{-1} = 0$, the value of β_0 is arbitrary.

Basic and Intermediate Questions on Quadrature Design and Analysis

Problem 3.8

Consider the quadrature formula

$$I_0(f) = (b-a)f(a) \approx \int_a^b f(x)dx = \mathcal{I}(f)$$

- What is the degree of exactness?
- What is the order of infinitesimal?

Problem 3.9

Consider the two quadrature formulas

$$I_2(f) = \frac{2}{3}\left[2f(-1/2) - f(0) + 2f(1/2)\right]$$
$$I_4(f) = \frac{1}{4}\left[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)\right]$$

- What is the degree of exactness when $I_2(f)$ is used to approximate $\mathcal{I}(f;-1,1) = \int_{-1}^{1} f(x)dx$?
- What is the degree of exactness when $I_2(f)$ is used to approximate $\mathcal{I}(f;=0.5,0.5)) = \int_{-1/2}^{1/2} f(x)dx$?
- What is the degree of exactness when $I_4(f)$ is used to approximate $\mathcal{I}(f;-1,1) = \int_{-1}^{1} f(x)dx$?

Problem 3.10

In this problem we consider the numerical approximation of the integral

$$\mathcal{I}(f;-1,1) = \int_{-1}^{1} f(x)dx$$

with $f(x) = e^x$. In particular, we use a priori error estimation to choose a step size h for Newton Cotes or a number of points for a Gaussian integration method.

3.10.a

Consider the use of the composite Trapezoidal rule to approximate the integral $\mathcal{I}(f; -1, 1)$.

- Use the fact that we have an analytical form of $f(x)$ to estimate the error using the composite trapezoidal rule and to determine a stepsize h so that the error will be less than or equal to the tolerance 10^{-2}.
- Approximately how many points does your h require?

3.10.b

Consider the use of the Gauss-Legendre method to approximate the integral $\mathcal{I}(f; -1, 1)$. Use $n = 1$, i.e., two points x_0 and x_1 with weights γ_0 and γ_1.

- Use the fact that we have an analytical form of $f(x)$ to estimate the error that will result from using the two-point Gauss-Legendre method to approximate the integral.
- How does your estimate compare to the tolerance 10^{-2} used in the first part of the question?
- Recall that for $n = 1$ we have the Gauss Legendre nodes $x_0 \approx -0.5774$ and $x_1 \approx 0.5774$. Apply the method to approximate I and compare its error to your prediction. The true value is

$$\mathcal{I}(f; -1, 1) = \int_{-1}^{1} e^x dx \approx 2.3504$$

Problem 3.11

Let $U(x)$ and $V(x)$ be polynomials of degree n defined on $x \in [-1, 1]$. Let $x_j, \quad 0 \leq j \leq n$ and $\gamma_j, \quad 0 \leq j \leq n$ be the Gauss-Legendre quadrature points and weights. Finally, let $\ell_j(x), \quad 0 \leq j \leq n$ be the Lagrange characteristic interpolating polynomials defined with nodes at the Gauss-Legendre quadrature points.

Show that the following summation by parts formula holds:

$$\sum_{j=0}^{n} U'(x_j) V(x_j) \gamma_j = (U(1)V(1) - U(-1)V(-1)) - \sum_{j=0}^{n} U(x_j) V'(x_j) \gamma_j$$

Problem 3.12

Consider the quadrature formula

$$I_3(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$$

for the approximation of $\mathcal{I}(f) = \int_0^1 f(x)dx$, where $f \in C^4([0, 1])$.

Note the specific interval used here, i.e., $[0, 1]$ and that the quadrature rule uses the value of the derivative $f'(x)$ at $x = 0$.

Determine the coefficients α_j, for $j = 1, 2, 3$ in such a way that $I_3(f)$ has degree of exactness $s = 2$.

Also, for the resulting method, determine the leading term of the quature error, i.e., find C, d and r in $\mathcal{I}(f) - I_3(f) = Ch^r f^{(d)} + O(h^{r+1})$.

Problem 3.13

Consider numerically approximating the integral

$$\mathcal{I}(f) = \int_a^b f(x)dx$$

using the **open Newton-Cotes** with $n = 2$, i.e., 3 points

$$I_2^{(o)} = \frac{4}{3}\, h_2 \left[2f(x_0) - f(x_1) + 2f(x_2)\right].$$

(3.13.a) Determine C, d, and s in the error expression

$$\mathcal{I} - I_2^{(o)} = C(b-a)^d f^{(s)} + O\left((b-a)^{d+1}\right)$$

(3.13.b) Suppose $\mathcal{I}$ is numerically approximated using a composite method, $I_{c2}^{(o)}$, based on $I_2^{(o)}$ with m intervals each of size $H = (b-a)/m$. Determine C, d, and s in the error expression

$$\mathcal{I} - I_{c2}^{(o)} = C(b-a)H^d f^{(s)} + O\left(H^{d+1}\right)$$

(3.13.c) Suppose global step halving is used to define a coarse grid with m intervals of size H_c and a fine grid with $2m$ intervals of size $H_f = \alpha H_c$ for the composite method, $I_{c2}^{(o)}$, where $\alpha = 0.5$. Determine, per interval on the coarse grid, the number of function evaluations made on the coarse grid that can be reused on the fine grid.

(3.13.d) Determine the number of new function evaluations required per interval on the fine grid to generate $I_{c2}^{(o)}$ on the fine grid.

(3.13.e) Is there a step refinement $\alpha \neq 0.5$ that allows you to reuse all of the function evaluations from the coarse grid with interval size H_c on the fine grid with interval size $H_f = \alpha H_c$?