Study Questions Homework 2 Foundations of Computational Math 2 Spring 2025

Problem 2.1

Consider the data points

$$(x, y) = \{(0, 2),\ \ (0.5, 5),\ \ (1, 8)\}$$

2.1.a. Write the interpolating polynomial in Lagrange form for the given data.

2.1.b. Write the interpolating polynomial in Newtong form for the given data.

2.1.c. Verify the relationship between the divided differences and the coefficients in the Lagrange form given in Set 7 of the class notes (Theorem 7.1).

Problem 2.2

Assume you are given distinct points $x_0, \dots, x_n$ and, $p_n(x)$, the interpolating polynomial defined by those points for a function f.

2.2.a. If $p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$ is the Lagrange form show that

$$\sum_{i=0}^{n} \ell_i(x) = 1$$

2.2.b. Assume $x \neq x_i$ for $0 \leq i \leq n$ and show that the divided difference $f[x_0, \dots, x_n, x]$ satisfies

$$f[x_0, \dots, x_n, x] = \sum_{i=0}^{n} \frac{f[x, x_i]}{\prod_{j=0, j\neq i}^{n}(x_i - x_j)}$$

Problem 2.3

Use this divided difference table for this problem.

i	0		1		2		3		4		5
x_i	-1		0		2		4		5		6
f_i	13		2		-14		18		67		91
$f[-,-]$		-11		-8		16		49		24	
$f[-,-,-]$			1		6		11		$-25/2$		
$f[-,-,-,-]$				1		1		$-47/8$			
$f[-,-,-,-,-]$					0		$-55/48$				
$f[-,-,-,-,-,-]$						$-55/336$					

2.3.a

Use the divided difference information about the unknown function $f(x)$ and consider the unique polynomial, denoted $p_{1,5}(x)$, that interpolates the data given by pairs (x_1, f_1), (x_2, f_2), (x_3, f_3), (x_4, f_4) , and (x_5, f_5). Use two different sets of divided differences to express $p_{1,5}(x)$ in two distinct forms.

2.3.b

What is the significance of the value of 0 for $f[x_0, x_1, x_2, x_3, x_4]$?

2.3.c

Denote by $p_{0,4}(x)$, the unique polynomial, that interpolates the data given by pairs (x_0, f_0), (x_1, f_1), (x_2, f_2), (x_3, f_3), and (x_4, f_4) and recall the definition of $p_{1,5}(x)$ from part (a). Use the divided difference information about the unknown function $f(x)$ to derive error estimates for $f(x) - p_{1,5}(x)$ and $f(x) - p_{0,4}(x)$ for any $x_0 \leq x \leq x_5$.

Problem 2.4

(Quarteroni et al. text exercise 8.10.1 on page 375)

Given n distinct points $x_0, \ldots, x_n$, show that the associated Lagrange form functions $\ell_i(x)$, $i = 0, \ldots, n$ form a basis for the vector space, $\mathbb{P}_n$, of polynomials with degree less than or equal to n.

Problem 2.5

(Quarteroni et al. text exercise 8.10.8 on page 376)

Consider the Hermite-Birkhoff interpolating polynomial defined by interpolation conditions:

$$p_n^{(k)}(x_0) = f^{(k)}(x_0), \quad 0 \leq k \leq n$$

where the superscript denotes the order of the derivative. Show that $p_n(x)$ is equal to the order n Taylor expansion of $f(x)$ around x_0

$$q_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Problem 2.6

(Quarteroni et al. text exercise 8.10.4 on page 376)

Suppose $x_0 < x_1 < \ldots < x_n$ are equally space points with $x_{i+1} - x_i = h$. Recall that $\omega_{n+1} = (x - x_0)(x - x_1) \cdots (x - x_n)$ and $\|f(x)\|_\infty = \max_{[a,b]} |f(x)|$ for a given interval $[a, b]$.

Derive an estimate or bound of $\|\omega_{n+1}(x)\|_\infty$ on the interval $[x_0, x_n]$ for $n = 1$, and $n = 2$.

Problem 2.7

Consider a polynomial

$$p_n(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

$p_n(\gamma)$ can be evaluated using Horner's rule (written here with the dependence on the formal argument x more explicitly shown)

$c_n(x) = \alpha_n$
for $i = n-1 : -1 : 0$
    $c_i(x) = x c_{i+1}(x) + \alpha_i$
end

$p_n(x) = c_0(x)$

Note that when evaluating $x = \gamma$ the algorithm produces $n+1$ constants $c_0(\gamma), \ldots, c_n(\gamma)$ one of which is equal to $p_n(\gamma)$.

2.7.a

Suppose that Horner's rule is applied to evaluate $p_n(\gamma)$ and that the constants $c_0(\gamma), \ldots, c_n(\gamma)$ are saved. Show that

$$p_n(x) = (x-\gamma)q(x) + p_n(\gamma)$$
$$q(x) = c_1(\gamma) + c_2(\gamma)x + \cdots + c_n(\gamma)x^{n-1}$$

2.7.b

Suppose that Horner's rule, with labeling modified appropriately, is applied to evaluate $p_n(\gamma)$ and that the constants $c_0^{(1)}(\gamma), \ldots, c_n^{(1)}(\gamma)$ are saved to define $p_n(\gamma) - c_0^{(1)}(\gamma)$ and $q_{(1)}(x) = c_1^{(1)}(\gamma) + c_2^{(1)}(\gamma)x + \cdots + c_n^{(1)}(\gamma)x^{n-1}$. Suppose further that Horner's rule is applied to evaluate $q_{(1)}(\gamma)$ and that the constants $c_1^{(2)}(\gamma), \ldots, c_n^{(2)}(\gamma)$ are saved to define $q_{(1)}(\gamma) = c_1^{(2)}(\gamma)$ and $q_{(2)}(x) = c_2^{(2)}(\gamma) + c_3^{(2)}(\gamma)x + \cdots + c_n^{(2)}(\gamma)x^{n-2}$. This can continue until Horner's rule is applied to evaluate $q_{(n)}(\gamma) = c_n^{(n)}(\gamma)$ and $q_{(n+1)}(x) = 0$, i.e., there are no constants other than $c_n^{(n)}(\gamma)$ produced.

Show that

$$q_{(1)}(\gamma) = p_n'(\gamma)$$
$$q_{(2)}(\gamma) = p_n''(\gamma)/2$$
$$q_{(3)}(\gamma) = p_n'''(\gamma)/3!$$
$$\vdots$$
$$q_{(n-1)}(\gamma) = p_n^{(n-1)}(\gamma)/(n-1)!$$
$$q_{(n)}(\gamma) = p_n^{(n)}(\gamma)/n!$$

and therefore form the coefficients of the Taylor form of $p_n(x)$

$$p_n(x) = p_n(\gamma)+(x-\gamma)p_n'(\gamma)+\frac{(x-\gamma)^2}{2}p_n''(\gamma)+\frac{(x-\gamma)^3}{3!}p_n'''(\gamma)\cdots+\frac{(x-\gamma)^{n-1}}{(n-1)!}p_n^{(n-1)}(\gamma)+\frac{(x-\gamma)^n}{n!}p_n^{(n)}(\gamma)$$

Problem 2.8

The set of square integrable functions

$$\mathcal{L}^2[-1,1] = \{f(x),\ -1 \le x \le 1 \mid \int_{-1}^{1} f^2(x)dx < \infty\}$$

is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\ g(x)dx$$

and the associated induced norm. The space of polynomials with degree n or less, $\mathbb{P}_n$, is a finite dimensional subspace of $\mathcal{L}^2[-1,1]$ with basis $\{b_k\} = \{x^k\}$ with $0 \le k \le n$.

A basis can be problematic if there is wide variation in the norm of the vectors, $\|b_k\|$ or if the angles between b_k and b_j become small for various pairs of vectors.

2.8.a. Analyze the magnitudes of the monomial basis vectors.

2.8.b. Analyze the angles between the monomial basis vectors.

2.8.c. Discuss the results in terms of the robustness of the basis for representing polynomials.

Problem 2.9

Show that given a set of points

$$x_0,\ \ x_1, \ldots, x_n$$

a Leja ordering can be computed in $O(n^2)$ operations.

Problem 2.10

Let $f(x)$ be a smooth function and let $p_n(x)$ be a polynomial of degree n that satisfies the Hermite-Brikhoff interpolation conditions for the point x_0

$$\begin{aligned} p_n(x_0) &= f(x_0) \\ p_n'(x_0) &= f'(x_0) \\ p_n''(x_0) &= f''(x_0) \\ &\vdots \\ p_n^{(n)}(x_0) &= f^{(n)}(x_0). \end{aligned}$$

2.10.a. Construct the Newton form of $p_n(x)$ using the Newton divided difference table. Identify and explain any structure in the divided difference table.

2.10.b. Using the basis that arises from the Newton form of $p_n(x)$, derive linear equations that impose the Hermite-Birkhoff interpolation conditions and therefore define the divided differences.

2.10.c. Show that $p_n(x)$ is unique and that the coefficients determined by solving the linear system are the same as those determined by using the divided difference table.

Problem 2.11

(2.11.a) Determine the polynomial of minimal degree that matches the following conditions on f or show that it does not exist:

$$\begin{aligned} f(0) = 0, &\quad f'(0) = 1 \\ f(1) = 3, &\quad f'(1) = 6 \end{aligned}$$

(2.11.b) Determine the polynomial of minimal degree that matches the following conditions on f or show that it does not exist:

$$\begin{aligned} f(0) = 0, &\quad f'(0) = 0 \\ f(1) = 3, &\quad f'(1) = 6 \\ f(2) &= 1 \end{aligned}$$

(2.11.c) Determine the polynomial of minimal degree that matches the following conditions on f or show that it does not exist. (Note that this is not an Hermite-Birkhoff form of interpolation problem.)

$$\begin{gathered} f(0) = 3 \\ f'(0) = 5, \quad f'(1) = 10, \quad f'(2) = 10 \end{gathered}$$

Problem 2.12

Let $f(x) = \cos 8x$ on $0 \leq x \leq \pi$. Suppose $f(x)$ is to be approximated by a piecewise linear interpolating function, $g_1(x)$. The accuracy required is

$$\forall 0 \leq x \leq \pi, \quad |f(x) - g_1(x)| \leq 10^{-6}$$

Determine a bound on $h = x_i - x_{i-1}$ for uniformly spaced points that satisfies the required accuracy.

Problem 2.13

Suppose we want to approximate a function$f(x)$ on the interval $[a, b]$ with a piecewise quadratic interpolating polynomial, $g_2(x)$, with a constant spacing, h, of the interpolation points $a = x_0 < x_1 \ldots < x_n = b$. That is, for any $a \leq x \leq b$, the value of $f(x)$ is approximated by evaluating the quadratic polynomial that interpolates f at x_{i-1}, x_i, and x_{i+1} for some i with $x = x_i + sh$, $x_{i-1} = x_i - h$, $x_{i+1} = x_i + h$ and $-1 \leq s \leq 1$. (How i is chosen given a particular value of x is not important for this problem. All that is needed is the condition $x_{i-1} \leq x \leq x_{i+1}$.)

Suppose we want to guarantee that the **relative error** of the approximation is less than 10^{-d}, i.e., d digits of accuracy. Specifically,

$$\frac{|f(x) - g_2(x)|}{|f(x)|} \leq 10^{-d}.$$

(It is assumed that $|f(x)|$ is sufficiently far from 0 on the interval $[a, b]$ for relative accuracy to be a useful value.) Derive a bound on h that guarantees the desired accuracy and apply it to interpolating $f(x) = e^x \sin x$ on the interval $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$ with relative accuracy of 10^{-4}. (The sin is bounded away from 0 on this interval.)

Compare your predicted accuracy to the accuracy you achieve by forming $g_2(x)$ for h's that satisfy your bound and h's that do not.

Problem 2.14

Consider the following data

$$(x_0, f_0) = (1, 0), \quad (x_1, f_1) = (2, 2),$$
$$(x_2, f_2) = (4, 12), \quad (x_3, f_3) = (5, 21)$$

2.14.a. Determine the quadratic interpolating polynomial, $p_2(x)$, for points (x_0, f_0), (x_1, f_1), (x_2, f_2). Estimate $f(3)$ using $p_2(x)$.

2.14.b. Determine the quadratic interpolating polynomial, $\tilde{p}_2(x)$, for points (x_1, f_1), (x_2, f_2), (x_3, f_3). Estimate $f(3)$ using $\tilde{p}_2(x)$.

2.14.c. Estimate $f(3)$ using a cubic interpolating polynomial $p_3(x)$.

2.14.d. Estimate the errors $|f(3) - p_2(x)|$ and $|f(3) - \tilde{p}_2(x)|$ an use the estimates to determine a range of values in which you expect $f(3)$ to reside. How does the value of $p_3(3)$ relate to this interval?

2.14.e. Write the piecewise linear interpolant $g_1(x)$ that uses all of the data points in the form that specifies the set of intervals and the linear polynomial on each interval. Estimate $f(3)$ using $g_1(x)$.

2.14.f. Determine the cardinal basis form of $g_1(x)$. Verify that your cardinal basis form satisfies the interpolation constraints.