Written Problems Assignment 1 Foundations of Computational Math 1 Fall 2024

The solutions are due on Canvas by 11:59 PM on Monday September 16, 2024

Problem 1.1

Recall that $\mathcal{P}_n$, the set of polynomials of degree less than or equal to n, and the operation of polynomial addition is equivalent to the vector space $\mathbb{C}^{n+1}$. A particular representation of a polynomial $p(\tau)$ must be chosen to use this equivalence. Suppose the monomials x^k, $k = 0, \ldots, n$ are used, i.e., $p(\tau) = \alpha_0 + \alpha_1\tau + \cdots + \alpha_n\tau^n$ and therefore $p(\tau)$ is specified by a vector $a \in \mathbb{C}^{n+1}$ with $e_j^T a = \alpha_{j-1}$, $j = 1, \ldots, n+1$. Your solutions should use this representation when appropriate.

1.1.a. Show that the mapping from a polynomial $p(\tau) \in \mathcal{P}_n$ to its derivative with respect to τ, $p'(\tau) \in \mathcal{P}_n$ can be expressed as an $n+1 \times n+1$ matrix applied to a vector v, i.e., $v' = Dv$, where the vector $v \in \mathbb{C}^{n+1}$ represents $p(\tau)$ and the vector $v' \in \mathbb{C}^{n+1}$ represents $p'(\tau)$.

1.1.b. What is the null space $\mathcal{N}(D)$ and how does it relate to the derivatives of the polynomials?

1.1.c. Recall that the $n+1-$st derivative of a polynomial of degree less than or equal to n is identically 0. How is this reflected in the algebraic properties of D?

Problem 1.2

This problem considers alternate organizations of a matrix expression using associativity and distribution to influence the number of operations and work storage required to evaluate the expression.

Suppose the matrix $A \in \mathbb{R}^{n\times n}$ and vectors $x, y \in \mathbb{R}^{n\times 1}$ are given. These require $n^2 + 2n$ storage locations in order to specify the problem. Consider computing the matrix $M \in \mathbb{R}^{n\times n}$ given by

$$M = Axy^T(I + xy^T)A \in \mathbb{R}^{n\times n}.$$

This requires an additional n^2 storage locations to contain the result of the comptutation, for a total of $2n^2 + O(n)$ storage at least for this computation.

1.2.a. Suppose you are allowed to use $O(n^3)$ operations and additional matrices of work storage for intermediate results. Derive an algorithm using associativity and distribution that achieves this complexity and express it using mathematical pseudocode.

1.2.b. Determine the complexity by finding the coefficient in the operation count $Cn^3 + O(n^2)$.

1.2.c. Determine the amount of extra work storage required and express it in the form $\tilde{C}n^2 + O(n)$, i.e., this assumes that your work storage will include at least one matrix.

1.2.d. Now suppose you are allowed to only use $O(n^2)$ operations and some additional work storage for intermediate results. Derive an algorithm using associativity and distribution that achieves this complexity and express it using mathematical pseudocode.

1.2.e. Determine the complexity by finding the coefficient in the operation count $\hat{C}n^2 + O(n)$.

1.2.f. Determine the amount of extra work storage required and express it by determining the coefficient and power in the form $\tilde{C}n^k + O(n)$.

Problem 1.3

This problem considers the vector norms $\|.\|_1$,$\|.\|_2$,$\|.\|_\infty$ on $\mathbb{C}^n$

1.3.a

Draw or describe the unit spheres in $\mathbb{R}^2$

$$\mathcal{S}_1 = \{x \in \mathbb{R}^2 \ | \ \|x\|_1 = 1\}, \ \ \mathcal{S}_2 = \{x \in \mathbb{R}^2 \ | \ \|x\|_2 = 1\}, \ \ \mathcal{S}_\infty = \{x \in \mathbb{R}^2 \ | \ \|x\|_\infty = 1\}$$

1.3.b

Define the ray emanating from the origin in $\mathbb{R}^2$, $x(\sigma) = \sigma e_1 + \sigma e_2$, for $\sigma \geq 0$, i.e., the ray defined by $\xi_1 = \xi_2$ with $\xi_1 \geq 0$ and $\xi_2 \geq 0$. Consider the intersection points of this ray with each of the unit spheres in $\mathbb{R}^2$: $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_\infty$. How do the positions of these points support the development of an hypothesis for the final part of the question, i.e., how do they provide evidence that the inequality is not true or true?

1.3.c

Is the following inequality a true statement? If so prove it. If not provide a counterexample.

$$\forall x \in \mathbb{C}^n, \quad \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1.$$