Study Problems 6 Foundations of Computational Math 1 Fall 2024

Problem 6.1

Let $f(x) = x^3 - 3x + 1$. This polynomial has three distinct roots.

6.1.a Consider using the iteration function

$$\phi_1(x) = \frac{1}{3}(x^3 + 1)$$

Which, if any, of the three roots can you compute with $\phi_1(x)$ and how would you choose $x^{(0)}$ for each computable root?

6.1.b Consider using the iteration function

$$\phi_2(x) = \frac{3}{2}x - \frac{1}{6}(x^3 + 1)$$

Which, if any, of the three roots can you compute with $\phi_2(x)$ and how would you choose $x^{(0)}$ for each computable root?

6.1.c For each of the roots you identified as computable using either $\phi_1(x)$ or $\phi_2(x)$, apply the iteration to find the values of the roots. (You need not turn in any code, but using a simple program to do this is recommended.) Try multiple initial guesses for each root that satisfy the convergence conditions you developed in your analysis above and comment on your empirical observations when verifying the predicted behavior.

Problem 6.2

(This problem is in Quateroni, Sacco and Salieri, Numerical Mathematics, 2nd Edition, p. 283, Problem 2)

The Modified Newton Method is defined by the iteration

$$\phi(x) = x - m\frac{f(x)}{f'(x)}$$

where m is assumed known.

Suppose that f is a continuous function that is m times differentiable with $m \geq 1$ such that

$$f(\alpha) = f'(\alpha) = \ldots = f^{(m-1)}(\alpha) = 0 \quad \text{and} \quad f^{(m)}(\alpha) \neq 0.$$

Show that Modified Newton has order of convergence equal to 2, i.e., it is quadratically convergent.

Problem 6.3

(This problem is a subset of Quateroni, Sacco and Salieri, Numerical Mathematics, 2nd Edition, p. 283, Problem 6)

Consider the function $f(x) = x^2 - x - 2$ which has two roots $\alpha_1 = -1$ and $\alpha_2 = 2$ and the iterations

$$\phi_1(x) = x^2 - 2$$

$$\phi_2(x) = \sqrt{2 + x}$$

$$\phi_3(x) = -\sqrt{2 + x}$$

for computing one or both of the roots. Analyze the convergence properties of each in terms of intervals of convergence and rate of convergence.

Problem 6.4

(This problem is Quateroni, Sacco and Salieri, Numerical Mathematics, 2nd Edition, p. 284, Problem 8)

Propose and analyze at least two fixed-point iterations to find the root $\alpha \approx 0.5885.$ of $f(x) = e^{-x} - \sin x$.

Problem 6.5

6.5.a

Consider the polynomial

$$p(x) = x^3 - x - 5$$

The following three iterations can be derived from $p(x)$.

$$\phi_1(x) = x^3 - 5$$
$$\phi_2(x) = \sqrt[3]{x + 5}$$
$$\phi_3(x) = 5/(x^2 - 1)$$

The polynomial $p(x)$ has a root $r > 1.5$. Determine which of the $\phi_i(x)$ produce an iteration that converges to r.

6.5.b

Consider the function

$$f(x) = xe^{-x} - 0.06064$$

i Write the update formula for Netwon's method to find the root of $f(x)$.

ii $f(x)$ has a root of $\alpha = 0.06469263\ldots$. If you run Newton's method with $x_0 = 0$ convergence occurs very quickly, e.g., in double precision $|f(x_4)| \approx 10^{-10}$. However, if x_0 is large and negative or if x_0 is near 1 convergence is much slower until there is a very rapid improvement in accuracy in the last one or two steps. For example, if $x_0 = 0.98$ then

$$|f(x_{47})| \approx 0.6 \times 10^{-2}$$
$$|f(x_{49})| \approx 1.5 \times 10^{-5}$$
$$|f(x_{50})| \approx 2.8 \times 10^{-10}$$

Explain this behavior. You might find it useful to plot $f(x)$ and run a few examples of Newton's method.

Problem 6.6

Consider the fixed point iteration by the function

$$\phi(x) = x - \frac{(x^2 - 3)}{(x^2 + 2x - 3)}$$

The value $\alpha = \sqrt{3}$ is a fixed point for this iteration. Provide justification to all of your answers for the following:

6.6.a Show that there exists a nontrivial interval $\alpha - \delta < x < \alpha + \delta$ with $\delta > 0$ such that the iteration defined by $\phi(x)$ converges to α for any x_0 in the interval.

6.6.b Is the order of convergence on this interval linear ($p = 1$) or higher ($p \geq 2$)?

6.6.c Show that for $x > \alpha$ we have

$$\alpha < \phi(x) < x$$

and use this to explain the convergence of the iteration when $x_0 > \alpha$.

6.6.d Plot or otherwise enumerate values of the curves $y = \phi(x)$ and $y = x$ on the interval $0 < x < \beta$ for $\beta > \alpha$. Use the information to examine the behavior of the iteration for $0 < x_0 < \alpha$. For what subinterval, if any, do you expect convergence to α?

Problem 6.7

Consider the generic cubic polynomial with three distinct roots 0, $\rho > 0$ and $-\rho$ with the following form and properties:

$$f(x) = x(x-\rho)(x+\rho) = x^3 - \rho^2 x, \quad \rho > 0$$

$$f'(x) = 3x^2 - \rho^2 = 3(x^2 - \alpha^2) = 3(x-\alpha)(x+\alpha), \quad \alpha = \rho/\sqrt{3} > 0$$

- f is concave on $-\infty < x < -\alpha$.
- f is convex on $\alpha < x < \infty$
- f has the standard cubic form on $-\alpha < x < \alpha$

$$f(x) > 0 \quad \text{on} \quad -\alpha < x < 0$$

$$f(0) = 0$$

$$f(x) < 0 \quad \text{on} \quad 0 < x < \alpha$$

6.7.a Write the iteration $\phi(x)$ that defines Newton's method. For this problem, you may find it useful to rewrite the standard form into

$$\phi(x) = \gamma(x) \ x.$$

6.7.b Determine the point $0 < \xi < \alpha$ such that if $x_0 = \xi$ Newton's method cycles between two point (in exact arithmetic).

6.7.c Using the facts given above and any others you prove, describe the behavior of Newton's method on the following intervals and justify your claims:

$$x > \rho$$

$$\alpha < x < \rho$$

$$-\xi < x < \xi$$

Problem 6.8

Consider the polynomial

$$p(x) = x^3 - 3x - 3$$

and the following iterations

$$\phi_1(x) = \sqrt[3]{3x+3}$$

$$\phi_2(x) = \frac{x^3}{3} - 1$$

$$\phi_3(x) = \frac{2x^3+3}{3x^2-3}$$

6.8.a Determine which of the $\phi_i(x)$ have all of the roots of $p(x)$ as fixed points.

6.8.b $p(x)$ has a root $r > 1$. Determine which $\phi_i(x)$ converge to r for some nontrivial interval and the rates of convergence for the $\phi_i(x)$ that converge to r?