Study Problems 4 Foundations of Computational Math 1 Fall 2024

Problem 4.1

Suppose $\|v\|_\nu$ is a given vector norm on $\mathbb{R}^n$. The vector norm induces a matrix norm $\|A\|_\alpha$ on $\mathbb{R}^{n\times n}$ by a maximization-based definition.

4.1.a Show that the vector and induced matrix norms satisfy for any $A \in \mathbb{R}^{n\times n}$:

$$\forall v \in \mathbb{R}^n, \quad \|Av\|_\nu \leq \|A\|_\alpha \|v\|_\nu$$

4.1.b Show that for any pair of matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ the matrix norm satisfies

$$\|AB\|_\alpha \leq \|A\|_\alpha \|B\|_\alpha$$

Problem 4.2

Suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

4.2.a Show that A is nonsingular.

4.2.b Show that all of the eigenvalues of A are real and positive and that they have corresponding real eigenvectors.

4.2.c Show that for any $v_1 \in \mathbb{R}^n$ and $v_2 \in \mathbb{R}^n$, the function $\langle v_1, v_2\rangle_A = v_2^T A v_1$ is an inner product.

4.2.d It is known that if A is symmetric positive definite then it has a symmetric positive definite square root, $A^{1/2} \in \mathbb{R}^{n\times n}$, such that $A = A^{1/2}A^{1/2} = A^{1/2}A^{T/2} = \left(A^{1/2}\right)^2$. It is also known that an inner product induces a vector norm by $\|v\|_A^2 = \langle v, v\rangle_A = v^T A v$. Prove that $\|v\|_A$ is a vector norm using an alternate approach that relates $\|v\|_A$ to $\|\tilde{v}\|_2$ where $\tilde{v}$ is unique for each $v \in \mathbb{R}^n$ and then exploits that it is known $\|\tilde{v}\|_2$ is a vector norm, i.e. it satisfies the required properties.

Problem 4.3

Suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Recall, that A has several well known factorizations due to symmetry and positive definiteness.

1. $A = A^T$ implies that $A = Q\Lambda Q^T$ where $Q^TQ = QQ^T = I \in \mathbb{R}^{n\times n}$ and $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^{n\times n}$, i.e., Λ is a real diagonal matrix. (Schur Decomposition)

2. If A is also positive definite then $A = LDL^T$ where $L \in \mathbb{R}^{n\times n}$ is a unit lower triangular matrix and $D = diag(\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}^{n\times n}$ with $\delta_i > 0$.

3. If A is also positive definite then $A = \tilde{L}\tilde{L}^T$ where $L \in \mathbb{R}^{n\times n}$ is a nonsingular lower triangular matrix with $e_i^T \tilde{L} e_i > 0$. (Cholesky factorization).

4.3.a Show that if A is symmetric then $\|A\|_2 = \rho(A)$ where $\rho(A)$ is the spectral radius of A. (Note there is no assumption of positive definite in this item.)

4.3.b Show if A is symmetric then that $\|A^{-1}\|_2 = 1/|\lambda_{min}|$ where λ_{min} is the eigenvalue of A that has the minimal magnitude. (Note there is no assumption of positive definite in this item.)

4.3.c Show that if $C \in \mathbb{R}^{n\times n}$ is nonsingular then CC^T is a symmetric positive definite matrix.

4.3.d Show that if A is symmetric positive definite then it has a symmetric positive definite square root, $A^{1/2} \in \mathbb{R}^{n\times n}$, such that $A = A^{1/2}A^{1/2} = A^{1/2}A^{T/2} = \left(A^{1/2}\right)^2$.

4.3.e Show that if A is symmetric positive definite with maximum and minimum eigenvalues λ_{max} and λ_{min} then

$$\forall w \in \mathbb{R}^n, \ \ 0 < \lambda_{min} \le \frac{w^T A w}{w^T w} \le \lambda_{max}$$

Problem 4.4

Two matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are similar if there exists a nonsingular $M \in \mathbb{R}^{n\times n}$ such that

$$A = M^{-1}BM.$$

4.4.a Show that A and B have the same eigenvalues.

4.4.b If (λ, v) is an eigenvalue and eigenvector pair of A what is an associated eigenvalue and eigenvector pair $(\tilde{\lambda}, \tilde{v})$ of B?

4.4.c Suppose that $A \in \mathbb{R}^{n\times n}$ and $P \in \mathbb{R}^{n\times n}$ are both nonsingular matrices. Show that $G = I - P^{-1}A$ and $\tilde{G} = I - AP^{-1}$ are similar matrices.

4.4.d Suppose that $A \in \mathbb{R}^{n\times n}$ and $P \in \mathbb{R}^{n\times n}$ are both symmetric positive matrices. Show that $G = I - P^{-1}A$, $\tilde{G} = I - AP^{-1}$ $\hat{G} = I - \hat{A}$ are all similar matrices, where $\hat{A} = C^{-1}AC^{-T}$ and $P = CC^T$ for some nonsingular $C \in \mathbb{R}^{n\times n}$.

Problem 4.5

Consider solving a linear system $Ax = b$ where A is symmetric positive definite using steepest descent.

4.5.a

Suppose you use steepest descent without preconditioning. Show that the residuals, r_k and r_{k+1} are orthogonal for all k.

4.5.b

Suppose you use steepest descent with preconditioning. Are the residuals, r_k and r_{k+1} orthogonal for all k? If not is there any vector from step k that is guaranteed to be orthogonal to r_{k+1}?

Problem 4.6

Let $A = Q\Lambda Q^T$ be a symmetric positive definite matrix where Q is an orthogonal matrix and Λ is a diagonal matrix whose diagonal elements are positive and also are the eigenvalues of A. Define

$$\tilde{x} = Q^T x \quad \text{and} \quad \tilde{b} = Q^T b$$
$$Ax = b \quad \text{and} \quad \Lambda\tilde{x} = \tilde{b}$$

Given x_0 and $\tilde{x}_0$, define the sequence x_k as the sequence of vectors produced by steepest descent applied to $Ax = b$ and the sequence $\tilde{x}_k$ as the sequence of vectors produced by steepest descent applied to $\Lambda\tilde{x} = \tilde{b}$.

Let $e^{(k)} = x_k - x$ and $\tilde{e}^{(k)} = \tilde{x}_k - \tilde{x}$. Show that if $\tilde{x}_0 = Q^T x_0$ then

$$\|e^{(k)}\|_2 = \|\tilde{e}^{(k)}\|_2, \quad k > 0$$

$$\|r_k\|_2 = \|\tilde{r}_k\|_2, \quad k > 0.$$

Also, what is the relationship between the stepsizes α_k and $\tilde{\alpha}_k$ for the x_k and $\tilde{x}_k$ iterations respectively.

Problem 4.7

Let $A \in \mathbb{R}^{n\times k}$, $x \in \mathbb{R}^k$, and $b \in \mathbb{R}^n$ with the columns of A linearly independent and consider the linear least squares problem

$$\min_{x\in\mathbb{R}^k} \|b - Ax\|_2$$

4.7.a. Show that $N = A^T A \in \mathbb{R}^{k\times k}$ is a symmetric positive definite matrix

4.7.b. Suppose that n and k are both very large and that A is very sparse, i.e., a small number of nonzero elements much less than k is in each row. Show how you would use CG without preconditioning to solve for the solution of the least squares problem x_{min} in a computationally efficient manner. Comment on the complexity of one step of your algorithm in terms of order of operations (you need not worry about the multiplicative constant in the order expressions).

Problem 4.8

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite and define the A-norm using the A-inner product

$$\langle v_1, v_2\rangle_A = v_2^T A v_1$$

$$\|v\|_A^2 = \langle v, v\rangle_A.$$

Consider the linear system $Ax = b$ with solution $x_* = A^{-1}b$. Define the two functions from $\mathbb{R}^n$ to $\mathbb{R}$

$$E(x) = \|x - x_*\|_A^2, \quad f(x) = \frac{1}{2}x^T A x - x^T b$$

(4.8.a) Show that $E(x)$ and $f(x)$ have the same unique minimizer x_*.

(4.8.b) What are the gradients $\nabla E(x)$ and $\nabla f(x)$?