Study Problems 2 Foundations of Computational Math 1 Fall 2024

All of these problems are based on the material in the notes on LU factorization and lectures. They are all of the basic type that you should be able to reproduce after studying the solutions. They include a derivation of Gauss-Jordan factorization based on elementary transformations related to Gauss Transformations that is close to the algorithm often taught as a way of computing Reduced Row Echelon Form. Also discussed is the use of pivoting for efficiency rather than numerical stability or existence of the factorization; an incremenatal or bordering form of producing a factorization based on partitioning; growth of elements in the factoriztion; and a key additional algebraic assumption, symmetric positive definiteness of A.

Problem 2.1

Suppose you are computing a factorization of the $A \in \mathbb{C}^{n\times n}$ with partial pivoting and at the beginning of step i of the algorithm you encounter the the transformed matrix with the form

$$T^{-1}A = A^{(i-1)} = \begin{pmatrix} U_{11} & U_{12} \\ 0 & A_{i-1} \end{pmatrix}$$

where $U_{11} \in \mathbb{R}^{i-1\times i-1}$ and nonsingular, and $U_{12} \in \mathbb{R}^{i-1\times n-i+1}$ contain the first $i-1$ rows of U. Show that if the first column of A_{i-1} is all 0 then A must be a singular matrix.

Problem 2.2

Suppose you have the LU factorization of an $i \times i$ matrix $A_i = L_i U_i$ and suppose the matrix A_{i+1} is an $i+1 \times i+1$ matrix formed by adding a row and column to A_i, i.e.,

$$A_{i+1} = \begin{pmatrix} A_i & a_{i+1} \\ b_{i+1}^T & \alpha_{i+1,i+1} \end{pmatrix}$$

where a_{i+1} and b_{i+1} are vectors in $\mathbb{R}^i$ and $\alpha_{i+1,i+1}$ is a scalar.

2.2.a. Derive an algorithm that, given L_i, U_i and the new row and column information, computes the LU factorization of A_{i+1} **and identify the conditions under which the step will fail**.

2.2.b. What computational primitives are involved?

2.2.c. Show how this basic step could be used to form an algorithm that computes the LU factorization of an $n \times n$ matrix A.

Problem 2.3

Suppose that $A \in \mathbb{R}^{n\times n}$ is nonsingular and that $A = LU$ is its LU factorization. Give an algorithm that can compute, $e_i^T A^{-1} e_j$, i.e., the (i,j) element of A^{-1} in approximately $(n-j)^2 + (n-i)^2$ operations.

Problem 2.4

(Restated Golub and Van Loan 3rd Ed. p. 103 Problem P3.2.5.)

Define the elementary matrix $N_k^{-1} = I - y_k e_k^T \in \mathbb{R}^{n\times n}$, where $1 \leq k \leq n$ is an integer, $y_k \in \mathbb{R}^n$ and $e_k \in \mathbb{R}^n$ is the k-th standard basis vector. N_k^{-1} is a Gauss-Jordan transform if it is defined by requiring $N_k^{-1} v = e_k\, \nu_k$ for a particular given vector $v \in \mathbb{R}^n$ whose elements are denoted $\nu_j = e_j^T v$. For example, if $n = 6$ and $k = 3$ then

$$N_3^{-1} = \begin{pmatrix} 1 & 0 & * & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 & 0 \\ 0 & 0 & * & 0 & 1 & 0 \\ 0 & 0 & * & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \\ \nu_5 \\ \nu_6 \end{pmatrix}$$

where $*$ indicates a value that must be determined.

(2.4.a) Determine how to choose y_k and define N_k^{-1} given a vector $v \in \mathbb{R}^n$, i.e., determine the values of the elements of y_k in terms of the values of the elements of v so that $N_k^{-1} v = e_k\, \nu_k$. For $n = 6$ and $k = 3$ then

$$N_3^{-1} v = \begin{pmatrix} 1 & 0 & * & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 & 0 \\ 0 & 0 & * & 0 & 1 & 0 \\ 0 & 0 & * & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \\ \nu_5 \\ \nu_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \nu_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(2.4.b) Determine when N_k^{-1} exists and is nonsingular.

(2.4.c) Show how a series of N_k^{-1} can be used to transform a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ into a nonsingular diagonal matrix $D \in \mathbb{R}^{n\times n}$, i.e., all of the off-diagonal elements of D are 0 and all of the diagonal elements are nonzero. You may assume that A is such that all of the N_k^{-1} exist.

(2.4.d) Does the factorization that this transformation induces have any structure other than that in D?

Problem 2.5

Consider an $n \times n$ real matrix where

- $\alpha_{ij} = e_i^T A e_j = -1$ when $i > j$, i.e., all elements strictly below the diagonal are -1;
- $\alpha_{ii} = e_i^T A e_i = 1$, i.e., all elements on the diagonal are 1;
- $\alpha_{in} = e_i^T A e_n = 1$, i.e., all elements in the last column of the matrix are 1;
- all other elements are 0

For $n = 4$ we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

2.5.a. Compute the factorization $A = LU$ for $n = 4$ where L is unit lower triangular and U is upper triangular.

2.5.b. What is the pattern of element values in L and U for any n?

Problem 2.6

Suppose $A \in \mathbb{R}^{n \times n}$ is a nonsymmetric nonsingular matrix with the following nonzero pattern (shown for $n = 6$)

$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

Suppose A has an LU factorization that can be computed without partial or complete pivoting while being numerical reliable.

2.6.a. Suppose $A = LU$ is computed without any pivoting for this pattern of nonzero elements. Given that the number of operations in the algorithm is of the form $Cn^k + O(n^{k-1})$, where C is a constant independent of n and $k > 0$, what are C and k?

2.6.b. Now suppose you use pivoting to reduce complexity. Describe an algorithm that computes a factorization of a permuted A and then is used to solve $Ax = b$ as efficiently as possible.

2.6.c. Given that the number of operations in the algorithm is of the form $Cn^k + O(n^{k-1})$, where C is a constant independent of n and $k > 0$, what are C and k for the efficient algorithm? Do you expect a significant reduction in computational time compared to the first algorithm that does not pivot?

Problem 2.7

Consider a symmetric matrix A, i.e., $A = A^T$.

2.7.a. Consider the use of Gauss transforms to factor $A = LU$ where L is unit lower triangular and U is upper triangular. **You may assume that the factorization does not fail.** Show that $A = LDL^T$ where L is unit lower triangular and D is a matrix with nonzeros on the main diagonal. i.e., elements in positions (i, i), and zero everywhere else, by demonstrating that L and D can be computed by applying Gauss transforms appropriately to the matrix A.

2.7.b. For an arbitrary symmetric matrix the LDL^T factorization will not always exist due to the possibility of 0 in the (i, i) position of the transformed matrix that defines the i-th Gauss transform. Suppose, however, that A is a **positive definite** symmetric matrix, i.e., $x^T Ax > 0$ for any vector $x \neq 0$. Show that the diagonal element of the transformed matrix A that is used to define the vector l_i that determines the Gauss transform on step i, $M_i^{-1} = I - l_i e_i^T$, is always positive and therefore the factorization will not fail. Combine this with the existence of the LDL^T factorization to show that, in this case, the nonzero elements of D are in fact positive.