Study Problems 1 Foundations of Computational Math 1 Fall 2024

These are study questions and you need not submit solutions. Solutions will be posted on the class webpage. The questions span a range of complexity and topics and therefore contain some that are here only for providing more details on classic results mentioned in the notes and that you will encounter in the literature. For those, you are not expected to duplicate that effort on any graded activity for this class.

Sections 1 and 2 contain mostly problems that can be solved using basic knowledge. Section 3 contains problems with proofs that will be clear and based on techniques that you will understand after we finish that section of the notes. Section 4 contain proofs that are important to see but not in any way such that you would be expected to reproduce the logic or similar proofs in a graded manner for this class.

1 Basic Space and Computational Primitives

Problem 1.1

Consider the vector space $\mathbb{R}^4$

1.1.a. Specify a subspace of $\mathbb{R}^4$ with dimension 2 by giving a basis for the subspace.

1.1.b. Show that the basis for a subspace is not unique by giving another basis for the same subspace given in (1.1.a).

Problem 1.2

1.2.a. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ be nonsingular matrices. Show $(AB)^{-1} = B^{-1}A^{-1}$.

1.2.b. Suppose $A \in \mathbb{R}^{m\times n}$ with $m > n$ and let $M \in \mathbb{R}^{n\times n}$ be a nonsingular square matrix, i.e., $M^{-1} \in \mathbb{R}^{n\times n}$ exists. Show that $\mathcal{R}(A) = \mathcal{R}(AM)$ where $\mathcal{R}()$ denotes the range of a matrix.

Problem 1.3

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function, i.e.,

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$$

.

1.3.a. Suppose you are given a routine that returns $F(x)$ given any $x \in \mathbb{R}^n$. How would you use this routine to determine a matrix $A \in \mathbb{R}^{m\times n}$ such that $F(x) = Ax$ for all $x \in \mathbb{R}^n$?

1.3.b. Show A is unique.

Problem 1.4

Let $\mathcal{S}_1 \subset \mathbb{R}^n$ and $\mathcal{S}_2 \subset \mathbb{R}^n$ be two subspaces of $\mathbb{R}^n$.

1.4.a. Suppose $x_1 \in \mathcal{S}_1$, $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. $x_2 \in \mathcal{S}_2$, and $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. Show that x_1 and x_2 are linearly independent.

1.4.b. Suppose $x_1 \in \mathcal{S}_1$, $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. $x_2 \in \mathcal{S}_2$, and $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. Also, suppose that $x_3 \in \mathcal{S}_1 \cap \mathcal{S}_2$ and $x_3 \neq 0$, i.e., the intersection is not empty. Show that x_1, x_2 and x_3 are linearly independent.

Problem 1.5

Suppose $A \in \mathbb{R}^{m\times n}$, $m \geq n$ and $rank(A) = p \leq n$. Show that there exists $X \in \mathbb{R}^{m\times p}$ and $Y \in \mathbb{R}^{\times p}$ such that $rank(X) = rank(Y) = p$ and

$$A = XY^T$$

Hint: remember that the rank is the maximal number of linearly independent columns a matrix contains and they are a basis for the range of the matrix. Consider this when choosing factors.

Problem 1.6

1.6.a

Consider computing the matrix vector product $y = Tx$, i.e., you are given T and x and you want to compute y. Suppose further that the matrix $T \in \mathbb{R}^{n\times n}$ is tridiagonal with constant values on each diagonal. For example, if $n = 6$ then

$$\begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ \gamma & \alpha & \beta & 0 & 0 & 0 \\ 0 & \gamma & \alpha & \beta & 0 & 0 \\ 0 & 0 & \gamma & \alpha & \beta & 0 \\ 0 & 0 & 0 & \gamma & \alpha & \beta \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix}$$

(1.6.a) Write a simple loop-based psuedo-code that computes $y = Tx$ for such a matrix $T \in \mathbb{R}^n$.

(1.6.b) How many operations are required as a function of n?

(1.6.c) Describe your data structures and determine how many storage locations are required as a function of n?

(1.6.d) Suppose the diagonals of T are not constant. For example, if $n = 6$ then

$$\begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 & 0 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 & 0 & 0 \\ 0 & 0 & \gamma_4 & \alpha_4 & \beta_4 & 0 \\ 0 & 0 & 0 & \gamma_5 & \alpha_5 & \beta_5 \\ 0 & 0 & 0 & 0 & \gamma_6 & \alpha_6 \end{pmatrix}.$$

Modify your algorithm to handle non-constant diagonal form and discuss the modifications made to the data structures required by the modification.

1.6.e

Suppose your algorithm is to run on a computer with an architecture that prefers "pipelined" or "vectorizable" code. Such computations are characterized by innermost loops, in loop-based code, that have iterations that are vectorizable. Specifically, the output value computed on an iteration of the loop is **not needed** as an input variable for any other iteration of the same loop, i.e., the iterations are independent and can in fact be done simultaneously. An alternative useful characterization is the the computations specified by the inner loop of any nested loop can be expressed as a vector operation, e.g., component by component addition or multiplication of two vectors, scaling a vector by a scalar etc.

Additionally, the data accessed by the all of the iterations is mostly data that can be accessed by simply indexed array element and is stored in memory close to each other. For example, the consecutively indexed elements of a 1-D array, $ARRAY(I), ARRAY(I+1), ARRAY(I+2)\ldots$, are stored next to each other. A series of elements of such an array accessed with a large stride $ARRAY(I+1000), ARRAY(I+1001), ARRAY(I+1002)\ldots$, while still a vector access, would be less desireable. Data in 2-D arrays are stored based on the ordering specified by the language. For example, column-major ordering of a 3×2 array would be stored sequentially in memory with elements in the same column are near each other. So a 3×2 array would be ordered

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ARRAY(1,1), ARRAY(2,1), ARRAY(3,1), ARRAY(1,2), ARRAY(2,2), ARRAY(3,2)
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Does your algorithm for the non-constant diagonals form of T map well to such an architecture? If so explain why if not alter it so it does. Note you may use vector notation in your code to make this easier to express, e.g., accessing $ARRAY(I), ARRAY(I+1), ARRAY(I+2)$ could be expressed $ARRAY(I:I+2)$.

2 Norms and Some of Their Properties

Problem 1.7

This problem considers three basic vector norms: $\|.\|_1$,$\|.\|_2$,$\|.\|_\infty$.

1.7.a. Prove that $\|.\|_1$ is a vector norm.

1.7.b. Prove that $\|.\|_\infty$ is a vector norm.

1.7.c. Consider $\|.\|_2$.

(i) Show that $\|.\|_2$ is definite.

(ii) Show that $\|.\|_2$ is homogeneous.

(iii) Show that for $\|.\|_2$ the triangle inequality follows from the Cauchy inequality $|x^H y| \leq \|x\|_2\|y\|_2$. $\|x\|_2 = \|y\|_2 = 1$ and $x^H y = |x^H y|$, prove the Cauchy inequality holds for x and y.

Problem 1.8

Let $y \in \mathbb{R}^m$ and $\|y\|$ be any vector norm defined on $\mathbb{R}^m$. Let $x \in \mathbb{R}^n$ and A be an $m \times n$ matrix with $m > n$.

1.8.a. Show that the function $f(x) = \|Ax\|$ is a vector norm on $\mathbb{R}^n$ if and only if A has full column rank, i.e., $rank(A) = n$.

1.8.b. Suppose we choose $f(x)$ from part (1.8.a) to be $f(x) = \|Ax\|_2$. What condition on A guarantees that $f(x) = \|x\|_2$ for any vector $x \in \mathbb{R}^n$?

Problem 1.9

Suppose that $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ and let $E = uv^T$.

1.9.a. Show that $\|E\|_F = \|E\|_2 = \|u\|_2\|v\|_2$.

1.9.b. Show that $\|E\|_\infty = \|u\|_\infty\|v\|_1$.

Problem 1.10

Suppose $A \in \mathbb{C}^{m\times n}$. Consider the matrix norm $\|A\|$ induced by the two vector 1-norms $\|x\|_1$ and $\|y\|_1$ for $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ respectively,

$$\|A\| = \max_{\|x\|_1=1} \|Ax\|_1.$$

Is this induced norm the same as the matrix 1-norm defined by

$$\|A\|_1 = \max_{1\le i\le n} \|Ae_i\|_1?$$

If so prove it. If not give counterexample to disprove it.

Problem 1.11

Suppose $A \in \mathbb{C}^{m\times n}$. Consider the matrix norm $\|A\|$ induced by the two vector ∞-norms $\|x\|_\infty$ and $\|y\|_\infty$ for $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ respectively,

$$\|A\| = \max_{\|x\|_\infty=1} \|Ax\|_\infty.$$

Is this induced norm the same as the matrix ∞-norm defined by

$$\|A\|_\infty = \max_{1\le i\le m} \|e_i^H A\|_1?$$

If so prove it. If not give counterexample to disprove it.

Problem 1.12

Show that for any vector norm on $\mathbb{C}$,

$$\forall\, x,\ y \in \mathbb{C} \quad \|x-y\| \ge |\ \ \|x\| - \|y\|\ \ |$$

Problem 1.13

Show that if the integers p and q satisfy $p \le q$ then

$$\forall\, x \in \mathbb{C} \quad \|x\|_p \le \|x\|_q.$$

Problem 1.14

We have the following theorem relating inner products and norms.

Theorem 1. *Let $\mathcal{V}$ be a real vector space with a norm $\|v\|$.*

1. *If the norm $\|v\|$ satisfies the parallelogram law*

$$\forall x, \quad y \in \mathcal{V}, \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{1}$$

   for every pair of vectors $x \in \mathcal{V}$ and $y \in \mathcal{V}$ then the function

$$f(x,y) = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$$

   is an inner product on $\mathcal{V}$ and $f(x,x) = \|x\|^2$.

2. *If $\|v\|$ does not satisfy the parallelogram law* (1) *for every pair of vectors $x \in \mathcal{V}$ and $y \in \mathcal{V}$ then it is* **not** *generated by an inner product.*

Consider $\mathcal{V} = \mathbb{R}^2$.

1. Show that the vector p-norm with $p = 1$, $\|v\|_1$ is not generated by an inner product.
2. Show that the vector p-norm with $p = 3$, $\|v\|_3$ is not generated by an inner product.
3. Does this imply that the these two vector norms are not generated by an inner product for any $\mathcal{V} = \mathbb{R}^n$?

3 Triangular Matrices and Some Algorithms

Problem 1.15

Consider the matrix

$$L = \begin{pmatrix} \lambda_{11} & 0 & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{pmatrix}$$

Suppose that $\lambda_{11} \neq 0$, $\lambda_{33} \neq 0$, $\lambda_{44} \neq 0$ but $\lambda_{22} = 0$.

1.15.a. Show that L is singular.

1.15.b. Determine a basis for the nullspace $\mathcal{N}(L)$.

Problem 1.16

Let $n = 4$ and consider the lower triangular system $Lx = f$ of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_{21} & 1 & 0 & 0 \\ \lambda_{31} & \lambda_{32} & 1 & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

Recall, that the column-oriented algorithm can be derived from a factorization $L = L_1 L_2 L_3$ where L_i was an elementary unit lower triangular matrix associated with the i-th column of L.

Show that the row-oriented algorithm can be derived from a factorization of L of the form

$$L = R_2 R_3 R_4$$

where R_i is associated with the i-th row of L.

Problem 1.17

1.17.a

An **elementary unit upper triangular column form matrix** $U_i \in \mathbb{R}^{n \times n}$ is of the form

$$I + u_i e_i^T$$

where $u_i^T e_j = 0$ for $i \leq j \leq n$. This matrix has 1 on the diagonal and the nonzero elements of u_i appear in the i-th column above the diagonal.

For example, if $n = 3$ then

$$\begin{aligned} U_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \mu_{13} \\ \mu_{23} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \mu_{13} \\ 0 & 1 & \mu_{23} \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Let $U \in \mathbb{R}^{n \times n}$ be a unit upper triangular matrix. Show that the factorization

$$U = U_n U_{n-1} \cdots U_2,$$

where $U_i = I + u_i e_i^T$ and the nonzeros of u_i are the nonzeros in the i-th column of U above the diagonal, can be formed without any computations.

1.17.b

Now suppose that $U \in \mathbb{R}^{n\times n}$ is a upper triangular with diagonal elements μ_{ii}. Let $S_i \in \mathbb{R}^{n\times n}$ be a diagonal matrix with its i-th diagonal element $e_i^T S_i e_i = \mu_{ii}$ and all of the other diagonal elements $e_j^T S_i e_j = 1$ for $i \neq j$.

For example, if $n = 3$ then

$$\begin{aligned} S_1 &= \begin{pmatrix} \mu_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ S_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ S_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_{33} \end{pmatrix} \end{aligned}$$

Let $U_i = I + u_i e_i^T$ and the nonzeros of u_i be the nonzeros in the i-th column of U above the diagonal. (This implies that $U_1 = I$)

Show that

$$U = (S_n U_n)(S_{n-1}U_{n-1})\cdots(S_2U_2)(S_1U_1).$$

Note that U may be singular so some μ_{ii} may be 0. Therefore, a proof based on expressing the algorithm for the solution of $Ux = b$ in terms of U_i^{-1} and S_i^{-1}, as is done in the next part of the question, is not applicable.

1.17.c

From the factorization of the previous part of the problem, derive an algorithm to solve $Ux = b$ given U is an $n \times n$ nonsingular upper triangular matrix. Describe the basic computational primitives required.

Problem 1.18

Given the the scalars, $\gamma_0, \ldots, \gamma_n$ and $\beta_1, \ldots, \beta_n$, a first order linear recurrence that determines the values of the scalars $\alpha_0, \ldots, \alpha_n$ is defined as follows:

$$\begin{aligned} \alpha_0 &= \gamma_0 \\ \alpha_i &= \beta_i \alpha_{i-1} + \gamma_i, \quad i = 1, \cdots, n. \end{aligned}$$

1.18.a Show that this recurrence code solves a linear system of equations to get the values of $\alpha_0, \ldots, \alpha_n$. That is identify the form of a matrix L and a vector f such that

$$La = f, \quad a = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^{n+1}$$

1.18.b Comment on any structural properties of the matrix and how they are exploited in the algorithm.

1.18.c How many operations are required to solve the system using the algorithm you described?

1.18.d How much storage is required for the algorithm you described?

Problem 1.19

Consider the matrix-vector product $x = Lb$ where L is an $n \times n$ unit lower triangular matrix with **all** of its nonzero elements equal to 1. For example, if $n = 4$ then

$$x = Lb$$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

The vector x is called the scan of b. Show that, given the vector b, the vector x can be computed in $O(n)$ computations rather than the $O(n^2)$ typically required by a matrix vector product. Express your solution in terms of matrices and vectors.

4 Fundamental Norm and Inner Product Theorems

Problem 1.20

1.20.a

The following lemma is a classic result for convex and concave functions on $\mathbb{R}$.

Lemma (Jensen's Inequality)**.** *Suppose a function $f : \mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}$, scalars $0 \leq \lambda_k \leq 1$ such that $\sum_{k=1}^{n} \lambda_k = 1$, and scalars $\xi_k \in \mathcal{D}$, $k = 1, \ldots, n$ are given.*

If $f(\xi)$ is convex on $\mathcal{D}$ then

$$f\left(\sum_{k=1}^{n} \lambda_k \xi_k\right) \leq \sum_{k=1}^{n} \lambda_k f(\xi_k)$$

and $f(\xi)$ is concave on $\mathcal{D}$ then

$$f\left(\sum_{k=1}^{n} \lambda_k \xi_k\right) \geq \sum_{k=1}^{n} \lambda_k f(\xi_k).$$

Show that if $0 \leq \alpha_k \in \mathcal{R}$ and $0 < p_k \in \mathcal{Z}$ with $\sum_{k=1}^{n} 1/p_k = 1$, $k = 1, \ldots, n$ are given then

$$\prod_{k=1}^{n} \alpha_k \leq \sum_{k=1}^{n} \frac{1}{p_k}\, \alpha_k^{p_k}.$$

This is Generalized Young's Inequality. Young's Inequality is with $n = 2$.

1.20.b

Use the Generalized Young's Inequality to prove the inequality relating the arithmetic mean to the geometric mean of positive real numbers $0 < \xi_k < \infty$, $k = 1, \ldots, n$,

$$\left(\prod_{k=1}^{n} \xi_k\right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^{n} \xi_k.$$

Problem 1.21

Prove the following

Lemma (Hoelder's Inequality)**.** *If $a,\ b \in \mathbb{R}^n$ and positive integers p and q satisfy $p^{-1}+q^{-1} = 1$, equivalently $p = q/(q-1)$ or $q = p/(p-1)$, then*

$$a^T b \leq |a^T b| \leq \|a\|_p \|b\|_q.$$

Problem 1.22

Prove the following

Lemma (Minkowski's Inequality)**.** *If $x,\ y \in \mathbb{R}^n$ and $p > 0$ is an integer*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Problem 1.23

Theorem 2. *If $\mathcal{V}$ is a real vector space with a norm $\|v\|$ that satisfies the parallelogram law*

$$\forall x, \quad y \in \mathcal{V}, \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{2}$$

then the function

$$f(x,y) = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$$

is an inner product on $\mathcal{V}$ and $f(x,x) = \|x\|^2$.

This problem proves this theorem by a series of lemmas. Prove each of the following lemmas and then prove the theorem.

Lemma 3. $\forall x \in \mathcal{V}$

$$f(x,x) = \|x\|^2$$

Lemma 4. $\forall x, \quad y \in \mathcal{V}$ *$f(x,x)$ is definite and $f(x,y) = f(y,x)$, i.e., (f is symmetric)*

Lemma 5. *The following two "cosine laws" hold $\forall x, \quad y \in \mathcal{V}$:*

$$2f(x,y) = \|x+y\|^2 - \|x\|^2 - \|y\|^2 \tag{3}$$
$$2f(x,y) = -\|x-y\|^2 + \|x\|^2 + \|y\|^2 \tag{4}$$

Lemma 6. $\forall x, \quad y \in \mathcal{V}$:

$$|f(x,y)| \leq \|x\|\|y\| \tag{5}$$
$$f(x,y) = \gamma\|x\|\|y\|, \quad sign(\gamma) = sign(f(x,y)), \quad 0 \leq |\gamma| \leq 1 \tag{6}$$

Lemma 7. $\forall x, \quad y, \quad z \in \mathcal{V}$:

$$f(x+z,y) = f(x,y) + f(z,y)$$

Lemma 8. $\forall x, \quad y \in \mathcal{V}, \quad \alpha \in \mathbb{R}$

$$f(\alpha x, y) = \alpha f(x,y)$$