Graded Homework 7 Applied Linear Algebra 2 Spring 2024

The solutions are due on Canvas by 11:59 PM on Friday April 26, 2024

Open Notes, Reference Texts, and Solutions for Study Questions and Homework
No collaboration with class members and any others.
All source usage must be properly cited and explained. Simple duplication or quoting of a source of any type will not receive full credit.

Question	Points Possible	Points Awarded
1.	10	
2.	10	
3.	10	
4.	20	
Total Points	50	

Name:____

I affirm that I have neither given nor taken assistance from anyone.

Signature: ________________________

Problem 7.1

Let $A \in \mathbb{C}^{m\times n}$ have singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$. Two standard matrix norms are

$$\|A\|_2 = \max_{\|v\|_2=1} \|Av\|_2$$

$$\|A\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha_{ij}|^2.$$

7.1.a

Let $A \in \mathbb{C}^{m\times n}$ and assume $Q_1 \in \mathbb{C}^{m\times m}$ and $Q_2 \in \mathbb{C}^{n\times n}$ are unitary matrices. Show that

$$\|A\|_2 = \|Q_1 A Q_2\|_2 \text{ and } \|A\|_F = \|Q_1 A Q_2\|_F$$

7.1.b

Show that

$$\|A\|_2 = \sigma_1$$
$$\|A\|_F^2 = \sum_{j=1}^{n} \sigma_j^2.$$

Problem 7.2

Let $A \in \mathbb{R}^{m\times n}$ have rank k with $k \leq n$, i.e., it may be rank deficient. The pseudoinverse behaves much as the inverse for nonsingular matrices. To see this show the following identities are true (Stewart 73) and comment on the effect of rank deficiency on each:

7.2.a. $AA^\dagger A = A$

7.2.b. $A^\dagger AA^\dagger = A^\dagger$

7.2.c. $A^\dagger A = (A^\dagger A)^T$

7.2.d. $AA^\dagger = (AA^\dagger)^T$

7.2.e. If $A \in \mathbb{R}$ has orthonormal columns then $A^\dagger = A^T$. Why is this important for consistency with simpler forms of least squares problems that we have discussed?

Problem 7.3

Suppose $A \in \mathbb{R}^{m\times n}$ with $m \geq n$. Use the SVD of A to show that there exist $U \in \mathbb{R}^{m\times n}$ with $U^T U = I_n$ and a symmetric positive semidefinite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A = UP.$$

Recall a symmetric positive definite matrix is one where $M = M^T$ and for all $v \neq 0$, $v^T M v > 0$. M is positive semidefinite if there exists $z \neq 0$ with $z^T M z = 0$. So for all $v \neq 0$, $v^T M v \geq 0$.

Problem 7.4

Given that we know the SVD exists for any complex matrix $A \in \mathbb{C}^{m\times n}$, assume that $A \in \mathbb{R}^{m\times n}$ has rank k with $k \leq n$, i.e., A is real and it may be rank deficient, and show that the SVD of A is all real and has the form

$$A = U \begin{pmatrix} S \\ 0 \end{pmatrix} V^T = U_k \Sigma_k V_k^T$$

where $S \in \mathbb{R}^{n\times n}$ is diagonal with nonnegative entries,

$$\begin{gathered} U = \begin{pmatrix} U_k & U_{m-k} \end{pmatrix}, \quad U^T U = I_m \\ V = \begin{pmatrix} V_k & V_{n-k} \end{pmatrix}, \quad V^T V = I_n \\ U_k \in \mathbb{R}^{m\times k}, \quad \text{and} \quad V_k \in \mathbb{R}^{n\times k} \end{gathered}$$

Hint: Consider the relationship between the SVD and the symmetric eigenvalue decomposition.