Graded Homework 6 Applied Linear Algebra 2 Spring 2024

The solutions are due on Canvas by 11:59 PM on Friday April 19, 2024

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Question	Points Possible	Points Awarded
1.	10	
2.	10	
3.	10	
4.	20	
Total Points	50	

Name:____

I affirm that I have neither given nor taken assistance from anyone.

Signature: ________________________

Useful Facts

1. $A \in \mathbb{C}^{n\times n}$ is a normal matrix if and only if $AA^H = A^H A$ where H indicates Hermitian transpose of a matrix where if $A \in \mathbb{R}^{n\times n}$ then $A^H = A^T$ where T indicates the transpose of A.

2. If $A \in \mathbb{C}^{n\times n}$ is a normal matrix then there exist $Q \in \mathbb{C}^{n\times n}$ and $\Lambda \in \mathbb{C}^{n\times n}$ where Λ is a diagonal matrix and Q is a unitary matrix, i.e., $Q^H Q = QQ^H = I$, such that $A = Q\Lambda Q^H$.

3. If $A \in \mathbb{R}^{n\times n}$ is a normal matrix, i.e., $AA^T = A^T A$ then as above $A = Q\Lambda Q^H$ where Q and Λ are, in general, complex but for any truly complex eigenvalue $\lambda_j = \gamma_j + i\beta_j$ with $\beta_j \neq 0$ and $i = \sqrt{-1}$ the complex conjugate $\overline{\lambda}_j = \gamma_j - i\beta_j$ is also an eigenvalue.

4. If $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, i.e., $A = A^T$ then A is normal and, additionally, $A = Q\Lambda Q^T$ where $Q \in \mathbb{R}^{n\times n}$, $Q^T Q = QQ^T = I$ and $\Lambda \in \mathbb{R}^{n\times n}$ is a real diagonal matrix.

Problem 6.1

If $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix and $A = Q\Lambda Q^T$ is an eigendecompostion with real eigenvalues and real pairwise orthonormal eigenvectors. Let $Qe_j = q_j$ and $e_j^T \Lambda e_j = \lambda_j$, $j = 1, \dots, n$ be eigenpairs (λ_j, q_j).

Suppose $\lambda_1 = \lambda_2 = \mu$ and $\mu \neq \lambda_j$, $j = 3, \dots, n$, i.e. two eigenvalues are equal to a value different than all of the other eigenvalues. Show that there exist two vectors $\tilde{q}_1$ and $\tilde{q}_2$ that are not q_1 and q_2 such that $\tilde{q}_1$ and $\tilde{q}_2$ are eigenvectors of $\lambda_1 = \lambda_2 = \mu$ and the matrix

$$\tilde{Q} = \begin{bmatrix} \tilde{q}_1 & \tilde{q}_2 & q_3 & \dots & q_n \end{bmatrix}$$

is such that $A = \tilde{Q}\Lambda\tilde{Q}^T$ with $\tilde{Q}^T\tilde{Q} = I$.

In fact, there are many such pairs $\tilde{q}_1$ and $\tilde{q}_2$. Show a general relation ship between the two $n \times 2$ matrices

$$\begin{bmatrix} \tilde{q}_1 & \tilde{q}_2 \end{bmatrix} \text{ and } \begin{bmatrix} q_1 & q_2 \end{bmatrix}$$

that satisfy the requirement $A = \tilde{Q}\Lambda\tilde{Q}^T$ with $\tilde{Q}^T\tilde{Q} = I$.

Problem 6.2

If $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix and $A = Q\Lambda Q^T$ is an eigendecompostion with real eigenvalues and real pairwise orthonormal eigenvectors. Let $Qe_j = q_j$ and $e_j^T \Lambda e_j = \lambda_j$, $j = 1, \dots, n$ be eigenpairs (λ_j, q_j).

Note that for all (λ_j, q_j) it follows that

$$Aq_j - q_j\lambda_j = 0 \quad \text{and} \quad \frac{q_j^T A q_j}{q_j^T q_j} = q_j^T A q_j = \lambda_j.$$

6.2.a. Suppose $v \in \mathbb{R}^n$ is not an eigenvector for any λ_j, i.e.,

$$Av - v\lambda_j \neq 0, \quad j = 1, \ldots, n.$$

Consider the quotient

$$\mu = \frac{v^T A v}{v^T v}.$$

Show that μ is the scalar that makes (μ, v) as close as possible to an eigenpair of A in a least squares approximation sense of the definition of an eigenpair.

6.2.b. Show that μ can be expressed as a weighted average of the eigenvalues of A and explain the weights.

Problem 6.3

6.3.a. Determine two eigenpairs (λ_1, v_1) and (λ_2, v_2) of the matrix

$$M = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

where $\theta \in \mathbb{R}$ and $\theta \neq 0$.

6.3.b. A real matrix $S \in \mathbb{R}^{n \times n}$ is called a skew symmetric matrix if $-S = S^T = S^H$. Show that S is normal and its eigenvalues are purely imaginary or 0.

Problem 6.4

Let $T \in \mathbb{R}^{n \times n}$ be a symmetric tridiagonal matrix, i.e., $e_i^T T e_j = e_j^T T e_i$ and $e_i^T T e_j = 0$ if $j < i - 1$ or $j > i + 1$. Consider $T = QR$ where $R \in \mathbb{R}^{n \times n}$ is an upper triangular matrix and $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix.

Recall, the nonzero structure of R was derived in class and shown to be $e_i^T R e_j = 0$ if $j < i$ (upper triangular assumption) or if $j > i + 2$, i.e, nonzeros are restricted to the main diagonal and the first two superdiagonals.

6.4.a Show that Q has nonzero structure such that $e_i^T Q e_j = 0$ if $j < i - 1$, i.e., Q is upper Hessenberg.

6.4.b Show that $T_+ = RQ$ is a symmetric triagonal matrix.

6.4.c Prove the Lemma in the class notes that states that choosing the shift $\mu = \lambda$, where λ is an eigenvalue of T, results in a reduced T_+ with known eigenvector and eigenvalue.