Graded Homework 5 Applied Linear Algebra 2 Spring 2024

The solutions are due on Canvas by 11:59 PM on Tuesday April 2, 2024

Open Notes, Reference Texts, and Solutions for Study Questions and Homework
No collaboration with class members and any others.
All source usage must be properly cited and explained. Simple duplication or quoting of a source of any type will not receive full credit.

Question	Points Possible	Points Awarded
1.	10	
2.	20	
3.	10	
4.	10	
Total Points	50	

Name:____

I affirm that I have neither given nor taken assistance from anyone.

Signature: ________________________

Problem 5.1

Suppose the matrices $A \in \mathbb{R}^{n\times k}$, $x \in \mathbb{R}^k$, $V_s \in \mathbb{R}^{k\times s}$, $n > k > s+1$, with the columns of A linearly independent, and the columns of $V_s = \begin{bmatrix} v_1 & v_2 & \dots & v_s \end{bmatrix}$ also linearly independent.

5.1.a Consider the constrained linear least squares problem,

$$\min_{x\in x_0+\mathcal{R}(V_s)} \|b - Ax\|_2$$

where $x_0 \in \mathbb{R}^k$ and $b \in \mathbb{R}^n$ are given. (The constraint set contains vectors of the form $x = x_0 + v$, $v \in \mathcal{R}(V_s)$). Determine a system of equations that determine the unique solution $x^* = x_0 + V_s c_s^*$ where $c_s^* \in \mathbb{R}^s$.

5.1.b Now suppose a column is added to V_s to define $V_{s+1} = \begin{bmatrix} v_1 & v_2 & \dots & v_s & v_{s+1} \end{bmatrix}$ so that the columns of V_{s+1} are also linearly independent. Determine a system of equations that determine the unique solution $\tilde{x}^* = x_0 + V_{s+1}c_{s+1}^*$, where $c_{s+1}^* \in \mathbb{R}^{s+1}$, to the modified linear least squares problem

$$\min_{x\in x_0+\mathcal{R}(V_{s+1})} \|b - Ax\|_2.$$

5.1.c Give sufficient conditions on the columns of V_{s+1} so that the two solutions are related by

$$c_{s+1}^* = \begin{pmatrix} c_s^* \\ \gamma_{s+1}^* \end{pmatrix}$$

$$\tilde{x}^* = x_0 + V_{s+1}c_{s+1}^* = x^* + v_{s+1}\,\gamma_{s+1}^*$$

Hint: Consider the normal equations for the problems and then exploit the definition of V_{s+1} in terms of V_s and v_{s+1} to examine the block structure of the matrices and vectors in the normal equations.

Problem 5.2

5.2.a

If CG is used to solve $Ax = b$ where A is symmetric positive definite then the iterates and errors have the form

$$x_k = x_0 + \alpha_0 d_0 + \alpha_1 d_1 + \ldots + \alpha_{k-1} d_{k-1} = x_{k-1} + \alpha_{k-1} d_{k-1}$$

$$e^{(k)} = x^* - x_k, \quad x^* = A^{-1} b$$

$$e^{(0)} = \alpha_0 d_0 + \alpha_1 d_1 + \ldots + \alpha_{n-1} d_{n-1}, \quad \alpha_i = \frac{\langle e^{(0)}, d_i \rangle_A}{\langle d_i, d_i \rangle_A}$$

$$\langle d_i, d_j \rangle_A = d_i^T A d_j = 0 \text{ for } i \neq j, \quad \langle d_i, d_i \rangle_A = d_i^T A d_i = \|d_i\|_A^2 \neq 0$$

i.e., the vectors $\{d_0, \ldots, d_{n-1}\}$ are A-orthogonal.

It can be shown that taking an arbitrary x_0 and $d_0 = r_0 = b - Ax_0$ that we have the spaces $\mathcal{S}_k$ for $k = 0, \ldots, n-1$ with multiple bases and satisfying the conditions

$$\mathcal{S}_k = span\left[d_0, d_1, \ldots, d_{k-1}, d_k\right] = span\left[d_0, d_1, \ldots, d_{k-1}, r_k\right] = span\left[r_0, r_1, \ldots, r_{k-1}, r_k\right]$$

$$r_k^T d_j = 0, \quad j = 0, \ldots, k-1$$
$$r_i^T r_j = 0, \quad i \neq j, \quad 0leqi, j \leq n-1$$

$$x_{k+1} = x_0 + z_k = x_k + \alpha_k d_k, \quad z_k \in \mathcal{S}_k.$$

It is straightforward to show that for CG we have

$$r_1^T d_0 = r_1^T r_0 = 0$$

$$span\left[d_0, d_1\right] = span\left[r_0, r_1\right] = span\left[r_0, Ar_0\right].$$

Use the definitions and properties of CG given above and assume the induction hypothesis,

$$\mathcal{S}_{k-1} = span\left[r_0, Ar_0, \ldots, A^{k-2} r_0, A^{k-1} r_0\right]$$

to show that

$$\mathcal{S}_k = span\left[r_0, Ar_0, \ldots, A^{k-1} r_0, A^k r_0\right].$$

Hint: Consider the recurrence used in the efficient CG implementation to update r_{k-1} to r_k which relates r_{k-1}, r_k, d_{k-1} and A.

5.2.b

Show that x_k generated by CG satisfies

$$\|e^{(k)}\|_A^2 \leq \min_{x \in x_0 + \mathcal{S}_{k-1}} \|x^* - x\|_A^2.$$

(In fact, for CG it is a strict inequality but you need not prove that.)

Problem 5.3

Suppose A is symmetric positive definite matrix and the system $Ax = b$ with solution $x^* = A^{-1}b$ is to be solved by Steepest Descent and CG. An approximation of x^* , denoted v, is said to be accurate to d decimal digits if

$$\frac{\|x^* - v\|_A}{\|x^*\|_A} \leq 10^{-d}$$

where accuracy is measured using the A-norm in this case.

5.3.a. Suppose A is symmetric positive definite with a condition number of 10. Determine an expression for a lower bound on the number of iterations of Steepest Descent would be required to guarantee 6 places of accuracy in the solution of $Ax = b$ assuming that x_0 was accurate to 3 decimal digits?

5.3.b. Suppose all you know about A is its condition number. Would you expect Conjugate Gradient to be guaranteed to achieve the same accuracy as Steepest Descent in fewer steps than the the number you determined for the previous part of the question? If so what is the relationship between the two number of steps? If not, why not?

5.3.c. What other information about A would you want to know to show that the number of steps required by Conjugate Gradient to guarantee a given accuracy is less than the number of steps based on only the condition number?

Problem 5.4

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix, $C \in \mathbb{R}^{n \times n}$ be a symmetric nonsingular matrix, and $b \in \mathbb{R}^n$ be a vector. The matrix $M = C^2$ is therefore symmetric positive definite. Also, let $\tilde{A} = C^{-1}AC^{-1}$ and $\tilde{b} = C^{-1}b$.

The preconditioned Steepest Descent algorithm to solve $Ax = b$ is:

A, M are symmetric positive definite
x_0 arbitrary; $r_0 = b - Ax_0$; solve $Mz_0 = r_0$

do $k = 0, 1, \ldots$ until convergence

$$
\begin{aligned}
& w_k = A z_k \\
& \alpha_k = \frac{z_k^T r_k}{z_k^T w_k} \\
& x_{k+1} \leftarrow x_k + z_k \alpha_k \\
& r_{k+1} \leftarrow r_k - w_k \alpha_k \\
& \text{solve } M z_{k+1} = r_{k+1}
\end{aligned}
$$

end

The Steepest Descent algorithm to solve $\tilde{A}\tilde{x} = \tilde{b}$ is:

$\tilde{A}$ is symmetric positive definite
$\tilde{x}_0$ arbitrary; $\tilde{r}_0 = \tilde{b} - \tilde{A}\tilde{x}_0$; $\tilde{v}_0 = \tilde{A}\tilde{r}_0$

do $k = 0, 1, \ldots$ until convergence

$$
\begin{aligned}
& \tilde{\alpha}_k = \frac{\tilde{r}_k^T \tilde{r}_k}{\tilde{r}_k^T \tilde{v}_k} \\
& \tilde{x}_{k+1} \leftarrow \tilde{x}_k + \tilde{r}_k \tilde{\alpha}_k \\
& \tilde{r}_{k+1} \leftarrow \tilde{r}_k - \tilde{v}_k \tilde{\alpha}_k \\
& \tilde{v}_{k+1} \leftarrow \tilde{A}\tilde{r}_{k+1}
\end{aligned}
$$

end

Show that given the appropriate consistency between initial guesses the preconditioned steepest descent recurrences to solve $Ax = b$ can be derived from the steepest descent recurrences to solve $\tilde{A}\tilde{x} = \tilde{b}$.