Graded Homework 4 Applied Linear Algebra 2 Spring 2024

The solutions are due on Canvas by 11:59 PM on Monday March 25, 2024

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Question	Points Possible	Points Awarded
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
Total Points	50	

Name:____

I affirm that I have neither given nor taken assistance from anyone.

Signature: ________________________

Note that throughout this assignment, Steepest Descent refers to the algorithm defined on slide 19 of Set 6 of the class notes and the general descent algorithm of which Steepest Descent is a special case is defined on slide 31 of Set 6 of the class notes.

Problem 4.1

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite and define the A-norm using the A-inner product

$$\langle v_1, v_2 \rangle_A = v_2^T A v_1$$

$$\|v\|_A^2 = \langle v, v \rangle_A.$$

Consider the linear system $Ax = b$ with solution $x_* = A^{-1}b$. Define the two functions from $\mathbb{R}^n$ to $\mathbb{R}$

$$E(x) = \|x - x_*\|_A^2, \quad f(x) = \frac{1}{2}x^T A x - b^T x$$

(4.1.a) Show that $E(x)$ and $f(x)$ have the same unique minimizer x_*.

(4.1.b) If $Ax = b$ is solved using the general descent method the stepsize α_k, used in $x_{k+1} = x_k + \alpha_k p_k$, is defined in terms of p_k, r_k and A. Show that α_k is the solution of a $n \times 1$-dimensional minimization problem of the form

$$\min_{\alpha\in\mathbb{R}} \|v_1^{(k)} - v_2^{(k)}\alpha\|^2$$

expressed using its normal equations. In your solution, identify the vector norm used to define the $n \times 1$-dimensional minimization problem, give $v_1^{(k)}$ and $v_2^{(k)}$, and show how α_k arises from the associated normal equations.

Problem 4.2

Let $A = Q\Lambda Q^T$ be a symmetric positive definite matrix where Q is an orthogonal matrix and Λ is a diagonal matrix whose diagonal elements are positive and also are the eigenvalues of A. Define

$$\tilde{x} = Q^T x \quad \text{and} \quad \tilde{b} = Q^T b$$
$$Ax = b \quad \text{and} \quad \Lambda\tilde{x} = \tilde{b}$$

Given x_0 and $\tilde{x}_0$, define the sequence x_k as the sequence of vectors produced by steepest descent applied to $Ax = b$ and the sequence $\tilde{x}_k$ as the sequence of vectors produced by steepest descent applied to $\Lambda\tilde{x} = \tilde{b}$.

Let $e^{(k)} = x_k - x$ and $\tilde{e}^{(k)} = \tilde{x}_k - \tilde{x}$. Show that if $\tilde{x}_0 = Q^T x_0$ then

$$\|e^{(k)}\|_2 = \|\tilde{e}^{(k)}\|_2, \quad k > 0$$

$$\|r_k\|_2 = \|\tilde{r}_k\|_2, \quad k > 0.$$

Also, what is the relationship between the stepsizes α_k and $\tilde{\alpha}_k$ for the x_k and $\tilde{x}_k$ iterations respectively.

Problem 4.3

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix with eigendecomposition $A = Q\Lambda Q^T$ where Q is an orthogonal matrix and Λ is a diagonal matrix whose diagonal elements are positive and also are the eigenvalues of A. Consider solving the linear system $Ax = b$ with solution $x_* = A^{-1}b$. using the general descent method.

(4.3.a) Show that for the choice of stepsize α_k used in the method we have $r_{k+1}^T p_k = 0$, i.e., $r_{k+1} \perp p_k$ in the Euclidean inner product, where $r_{k+1} = b - Ax_{k+1}$ is the residual vector for x_{k+1}.

(4.3.b) Suppose we take a direction vector p_k such that $p_k \perp r_k$, where $r_k = b - Ax_k$ is the residual vector for x_k. How does this affect the iteration?

(4.3.c) A matrix polynomial of degree $k+1$ can be defined as $P_{k+1}(A) = \nu_0 I + \nu_1 A + \cdots + \nu_k A^k + \nu_{k+1} A^{k+1}$ where the ν_i are real scalars. When analyzing iterative methods for linear systems the matrix polynomial can often be expressed in the more specific product form of degree $k+1$

$$P_{k+1}(A) = \prod_{i=0}^{k} (I - \gamma_i A) \tag{1}$$

where the γ_i are real scalars. Consider solving $Ax = b$ using the Steepest Descent method, i.e., the general descent method with $p_k = r_k$. Show that the residual at step $k+1$, $r_{k+1} = b - Ax_{k+1}$ can be written as

$$r_{k+1} = P_{k+1}(A) r_0$$

where $r_0 = b - Ax_0$ and $P_{k+1}(A)$ has the product form of (1). Be specific about relating the γ_i to parameters in the Steepest Descent sequence.

(4.3.d) Assuming $\tilde{x}_k = Q^T x_k$, $k \geq 0$ and $\tilde{b} = Q^T b$, what matrix polynomial relates $\tilde{r}_{k+1} = \tilde{b} - \Lambda \tilde{x}_{k+1}$ and $\tilde{r}_0$ for the Steepest Descent method?

Problem 4.4

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite tridiagonal matrix, i.e., its elements are 0 when not on the main diagonal or first superdiagonal or first subdiagonal. For $n = 6$, A would have the form

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & 0 & 0 \\ 0 & \alpha_{32} & \alpha_{33} & \alpha_{34} & 0 & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & 0 \\ 0 & 0 & 0 & \alpha_{54} & \alpha_{55} & \alpha_{56} \\ 0 & 0 & 0 & 0 & \alpha_{65} & \alpha_{66} \end{pmatrix}$$

where $\alpha_{ij} = \alpha_{ji}$. Consider solving the linear system $Ax = b$ with solution $x_* = A^{-1}b$. using the general descent method.

Determine the computational complexity, i.e., what are the number of storage locations and the number of computations, for the method. Be sure to give the numbers for each major computation done in each iteration and for the matrix and any vectors required. Express the totals as

$$Cn^d + O(n^{d-1}) \ \textit{computations} \ \text{and} \ \tilde{C}n^{\tilde{d}} + O(n^{\tilde{d}-1}) \ \textit{locations}.$$

Problem 4.5

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix with eigendecomposition $A = Q\Lambda Q^T$ where Q is an orthogonal matrix and Λ is a diagonal matrix whose diagonal elements are positive and also are the eigenvalues of A. Consider solving the linear system $Ax = b$ with solution $x_* = A^{-1}b$. using the Steepest Descent method.

(4.5.a) Suppose the n eigenvalues of A all have the same value, i.e., $\lambda_{1,1} = \lambda_{2,2} = \ldots = \lambda_{n,n} = \mu > 0$. What behavior does this cause for the iteration from the Steepest Descent method for all $x_0 \in \mathbb{R}^n$?

(4.5.b) Now suppose the n eigenvalues of A on take on two distinct values, i.e., $\lambda_{1,1} = \lambda_{2,2} = \ldots = \lambda_{s,s} = \mu_1 > 0$ and $\lambda_{s+1,s+1} = \lambda_{s+2,s+2} = \ldots = \lambda_{n,n} = \mu_2 > 0$ with $\mu_1 \neq \mu_2$. Does the behavior you identified when all eigenvalues had the same value still occur? Justify your answer.

(4.5.c) For the situation where $\mu_1 \neq \mu_2$ are the only values taken on by the $\lambda_{i,i}$, relate the stepsize α_k used to compute $x_{k+1} = x_k + r_k\alpha_k$ in the Steepest Descent method to the μ_j and the residual vector r_k.