Graded Homework 3 Applied Linear Algebra 2 Spring 2024

The solutions are due on Canvas by 11:59 PM on Friday March 8, 2024

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Question	Points Possible	Points Awarded
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
9.	10	
10.	10	
Total Points	100	

Name:____

I affirm that I have neither given nor taken assistance from anyone.

Signature: ________________________

Problem 3.1

If $S \in \mathbb{R}^{n\times n}$ is symmetric, i.e., $S = S^T$ then there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$, i.e, $e_i^T \Lambda e_j = 0$ when $i \neq j$ and $e_i^T \Lambda e_i = \lambda_{ii}$ may or may not be 0, such that

$$S = Q\Lambda Q^T.$$

3.1.a. Show that λ_{ii} and $q_i = Qe_i$ is an eigenvalue and eigenvector pair for $i = 1, \ldots, n$.

3.1.b. Using the factorization above, show that if S is symmetric positive definite matrix then $\lambda_{ii} > 0$ for $i = 1, \ldots, n$.

Problem 3.2

3.2.a. Suppose the matrix $Q \in \mathbb{C}^{n\times n}$ is a unitary matrix, i.e, its columns are such that $\|Qe_i\|_2 = 1$ and $q_i^H q_j = 0$ when $i \neq j$. If $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ are an eigenvalue and eigenvector of Q, what can you say about $|\lambda|$?

3.2.b. A matrix $A \in \mathbb{C}^{n\times n}$ is nilpotent if $A^k = 0$ for some integer $k > 0$. Prove that the only eigenvalue of a nilpotent matrix is 0. (Hint: consider an argument based on contradiction.)

Problem 3.3

Suppose you are given a symmetric matrix $A \in \mathbb{R}^{n\times n}$ and consider the computation of the matrix-vector product $v \leftarrow Au$ where $u \in \mathbb{R}^n$ is given and $v \in \mathbb{R}^n$ is computed.

3.3.a Since the matrix is symmetric, there are only $n(n+1)/2$ elements that are free to choose while the others are set due to symmetry. Describe a data structure that would only store $n(n+1)/2$ values that specify A.

3.3.b Give the mappings from mathematical quantities such as $\alpha_{ij} = e_i^T A e_j$, $\nu_i = e_i^T v$, and $\mu_i = e_i^T u$ to the data structures you use for A, v, and u.

3.3.c Describe an algorithm using pseudo-code that uses your data structure to implement the computation of the matrix-vector product, $Au \to v$, given A and u. Make sure you point out all of the relevant features that influence efficiency.

When describing the algorithm use Matlab coding forms as shown in the posted solution to Program 1.

Problem 3.4

3.4.a

Suppose you are given the nonsingular lower triangular matrix $A \in \mathbb{R}^{n\times n}$ where the magnitude of the diagonal elements are maximal in the associated columns. For example, for $n = 6$

$$A = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 3 & 10 & 0 & 0 & 0 & 0 \\ 1 & 3 & 10 & 0 & 0 & 0 \\ 2 & 1 & 3 & 10 & 0 & 0 \\ 5 & 2 & 1 & 3 & 10 & 0 \\ 4 & 5 & 2 & 1 & 3 & 10 \end{pmatrix}.$$

The fact that the values are constant along the diagonals is not important for this problem.

i Suppose the factorization of A is done using a code that will use partial pivoting if required. What can be said about the structure of the resulting P, L and U factors for A such as the ones described above.

ii Verify your claims by computing the factorization $PA = LU$ and checking the values of each of the matrices for the example A given above.

3.4.b

Suppose you are given the nonsingular lower triangular matrix $A \in \mathbb{R}^{n\times n}$ where the magnitude of the diagonal elements are not maximal in the associated columns. For example, for $n = 6$

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 0 \\ 5 & 2 & 1 & 3 & 0 & 0 \\ 6 & 5 & 2 & 1 & 3 & 0 \\ 10 & 6 & 5 & 2 & 1 & 3 \end{pmatrix}.$$

The fact that the values are constant along the diagonals is not important for this problem.

i Suppose the factorization of A is done using a code that will use partial pivoting if required. What can be said about the structure of the resulting P, L and U factors for A such as the ones described above.

ii Verify your claims by computing the factorization $PA = LU$ and checking the values of each of the matrices for the example A given above.

Problem 3.5

Suppose you are given the nonsingular tridiagonal matrix $T \in \mathbb{R}^{n\times n}$ For example, if $n = 6$ then

$$\begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 & 0 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 & 0 & 0 \\ 0 & 0 & \gamma_4 & \alpha_4 & \beta_4 & 0 \\ 0 & 0 & 0 & \gamma_5 & \alpha_5 & \beta_5 \\ 0 & 0 & 0 & 0 & \gamma_6 & \alpha_6 \end{pmatrix}$$

3.5.a Suppose that computing the factorization of T **requires partial row pivoting to succeed. In fact, assume that every step requires an exchange, i.e.,** P_i **exchanges rows** i **and** $i+1$ **to bring a maximal magnitude element to the pivot position.** What can be said about the structure of the resulting P, L and U factors.

3.5.b Verify your claims by computing the factorization $PT = LU$ with partial row pivoting and checking the values of each of the matrices for

$$T = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 2 & 0 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix}.$$

The fact that the values on the diagonals are constant is of no importance for this problem.

3.5.c Describe data structures to be used in an efficient implementation? (You need not write the Matlab-like code for the algorithm.)

Problem 3.6

Let x and y be two vectors in $\mathbb{R}^n$.

3.6.a. Show that given x and y the value of $\|x - \alpha y\|_2$ is minimized when

$$\alpha_{min} = \frac{x^T y}{y^T y}$$

3.6.b. Show that $x = y\alpha_{min} + z$ where $y^T z = 0$, i.e., x is easily written as the sum of two orthogonal vectors with specifed minimization properties.

Problem 3.7

Consider a Householder reflector, H, in $\mathbb{R}^2$. Show that

$$H = \begin{pmatrix} -\cos(\phi) & -\sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

where ϕ is some angle.

Problem 3.8

3.8.a. Let $H = I + \alpha x x^T \in \mathbb{R}^{n \times n}$, where $\alpha = -2/\|x\|_2^2$, be a Householder reflector. Determine two distinct eigenvalues for H and associated eigenvectors.

3.8.b. Let $\gamma = \cos\theta$ and $\sigma = \sin\theta$ and consider the 2×2 matrix

$$M = \begin{pmatrix} \gamma & \sigma \\ -\sigma & \gamma \end{pmatrix}$$

Determine the eigenvalues and eigenvectors of M.

Problem 3.9

Let $A \in \mathbb{R}^{n \times k}$ have rank k, i.e., have k linearly independent columns. The linear least squares problem

$$\min_{x \in \mathbb{R}^k} \|b - Ax\|_2$$

has a unique solution x_{min} for any $b \in \mathbb{R}^n$. The mapping $b \mapsto x_{min}$ defines a linear transformation, $A^\dagger$, from $\mathbb{R}^n$ to $\mathbb{R}^k$ called the pseudoinverse.

The pseudoinverse for rectangular full column-rank matrices behaves much as the inverse for nonsingular matrices. To see this answer the following questions and show the following identities are true :

3.9.a. Use the normal equations to write $A^\dagger$ in terms of A.

3.9.b. If $A \in \mathbb{R}^{n \times n}$ what is $A^\dagger$?

3.9.c. $AA^\dagger A = A$

3.9.d. $A^\dagger A A^\dagger = A^\dagger$

3.9.e. $A^\dagger A = (A^\dagger A)^T$

3.9.f. $AA^\dagger = (AA^\dagger)^T$

3.9.g. If $A \in \mathbb{R}^{n \times k}$ has orthonormal columns then $A^\dagger = A^T$. Why is this important for consistency with simpler forms of least squares problems that we have discussed?

Problem 3.10

Any subspace $\mathcal{S}$ of $\mathbb{R}^n$ of dimension $k \leq n$ must have at least one orthogonal matrix $Q \in \mathbb{R}^{n \times k}$ with orthonormal columns such that $\mathcal{R}(Q) = \mathcal{S}$, The matrix $P = QQ^T$ is a projector onto $\mathcal{S}$, i.e., Px is the unique component of x contained in $\mathcal{S}$.

3.10.a. P is clearly symmetric, show that it is idempotent, i.e., $P^2 = P$.

3.10.b. Show that $\mathcal{R}(P) = \mathcal{S}$.

3.10.c. Show that if M is an idempotent symmetric matrix then it is a projector onto $\mathcal{R}(M)$.

3.10.d. Choose any subspace of $\mathbb{R}^3$ that has dimension 2 and construct two orthornormal bases, Q_1 and Q_2. Verify that $P = Q_1 Q_1^T = Q_2 Q_2^T$.