Graded Homework 2 Applied Linear Algebra 2 Spring 2024

The solutions are due on Canvas by 11:59 PM on Wednesday February 14, 2024

Problem 2.1

Consider computing the matrix vector product $y = Tx$, i.e., you are given T and x and you want to compute y. Suppose further that the matrix $T \in \mathbb{R}^{n \times n}$ is tridiagonal with constant values on each diagonal. For example, if $n = 6$ then

$$\begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ \gamma & \alpha & \beta & 0 & 0 & 0 \\ 0 & \gamma & \alpha & \beta & 0 & 0 \\ 0 & 0 & \gamma & \alpha & \beta & 0 \\ 0 & 0 & 0 & \gamma & \alpha & \beta \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix}$$

(2.1.a) Write a simple loop-based psuedo-code that computes $y = Tx$ for such a matrix $T \in \mathbb{R}^n$.

(2.1.b) How many operations are required as a function of n?

(2.1.c) Describe your data structures and determine how many storage locations are required as a function of n?

(2.1.d) Suppose the diagonals of T are not constant. For example, if $n = 6$ then

$$\begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 & 0 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 & 0 & 0 \\ 0 & 0 & \gamma_4 & \alpha_4 & \beta_4 & 0 \\ 0 & 0 & 0 & \gamma_5 & \alpha_5 & \beta_5 \\ 0 & 0 & 0 & 0 & \gamma_6 & \alpha_6 \end{pmatrix}.$$

Modify your algorithm to handle non-constant diagonal form and discuss the modifications made to the data structures required by the modification.

Problem 2.2

Suppose you are given a symmetric matrix $A \in \mathbb{R}^{n \times n}$ and consider the computation of the matrix-vector product $v \leftarrow Au$ where $u \in \mathbb{R}^n$ is given and $v \in \mathbb{R}^n$ is computed.

2.2.a Since the matrix is symmetric, there are only $n(n+1)/2$ elements that are free to choose while the others are set due to symmetry. Describe a data structure that would only store $n(n+1)/2$ values that specify A.

2.2.b Describe an algorithm using pseudo-code that uses your data structure to implement the computation of the matrix-vector product, $v \leftarrow Au$, given A and u. Make sure you point out all of the relevant features that influence efficiency.

Problem 2.3

Suppose that $A \in \mathbb{R}^{n\times n}$ is nonsingular and that $A = LU$ is its LU factorization. Give an algorithm that can compute, $e_i^T A^{-1} e_j$, i.e., the (i,j) element of A^{-1} in approximately $(n-j)^2 + (n-i)^2$ operations.

Problem 2.4

Consider an $n \times n$ real matrix where

- $\alpha_{ij} = e_i^T A e_j = -1$ when $i > j$, i.e., all elements strictly below the diagonal are -1;
- $\alpha_{ii} = e_i^T A e_i = 1$, i.e., all elements on the diagonal are 1;
- $\alpha_{in} = e_i^T A e_n = 1$, i.e., all elements in the last column of the matrix are 1;
- all other elements are 0

For $n = 4$ we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

2.4.a. Compute the factorization $A = LU$ for $n = 4$ where L is unit lower triangular and U is upper triangular.

2.4.b. What is the pattern of element values in L and U for any n?

Problem 2.5

Suppose you have the LU factorization of an $i \times i$ matrix $A_i = L_i U_i$ and suppose the matrix A_{i+1} is an $i+1 \times i+1$ matrix formed by adding a row and column to A_i, i.e.,

$$A_{i+1} = \begin{pmatrix} A_i & a_{i+1} \\ b_{i+1}^T & \alpha_{i+1,i+1} \end{pmatrix}$$

where a_{i+1} and b_{i+1} are vectors in $\mathbb{R}^i$ and $\alpha_{i+1,i+1}$ is a scalar.

2.5.a. Derive an algorithm that, given L_i, U_i and the new row and column information, computes the LU factorization of A_{i+1} **and identify the conditions under which the step will fail**.

2.5.b. What computational primitives are involved?

2.5.c. Show how this basic step could be used to form an algorithm that computes the LU factorization of an $n \times n$ matrix A.

Problem 2.6

Suppose you are computing a factorization of the $A \in \mathbb{C}^{n\times n}$ with partial pivoting and at the beginning of step i of the algorithm you encounter the the transformed matrix with the form

$$TA = A^{(i-1)} = \begin{pmatrix} U_{11} & U_{12} \\ 0 & A_{i-1} \end{pmatrix}$$

where $U_{11} \in \mathbb{R}^{i-1\times i-1}$ and nonsingular, and $U_{12} \in \mathbb{R}^{i-1\times n-i+1}$ contain the first $i-1$ rows of U. Show that if the first column of A_{i-1} is all 0 then A must be a singular matrix.

Problem 2.7

Suppose you are given a tridiagonal matrix $T \in \mathbb{R}^{n\times n}$ For example, if $n = 6$ then

$$\begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 & 0 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 & 0 & 0 \\ 0 & 0 & \gamma_4 & \alpha_4 & \beta_4 & 0 \\ 0 & 0 & 0 & \gamma_5 & \alpha_5 & \beta_5 \\ 0 & 0 & 0 & 0 & \gamma_6 & \alpha_6 \end{pmatrix}.$$

Assume that $T \in \mathbb{R}^{n\times n}$ is strictly diagonally dominant by columns and that the elements on the main diagonal are all positive, i.e., $\alpha_i > 0$ for $1 \le i \le n$.

(2.7.a) Determine the form of the matrix after one step of LU factorization without pivoting (which is not needed since diagonal dominance is assumed).

(2.7.b) How many operations were required to perform the single step?

(2.7.c) What is the structure of the active part of the matrix after one step, i.e., what is the Schur complement of T with respect to α_1?

(2.7.d) Suppose in addition it is assumed that $\beta_i \ge 0$ for $1 \le i \le n-1$ and $\gamma_i \ge 0$ for $2 \le i \le n$. What is the maximum growth of the elements in the active part of the matrix after one step relative to the elements in T?

(2.7.e) What does this one step growth imply about the Wilkinson growth factor after completing the LU factorization?

(2.7.f) Does your conclusion about the growth factor change if the assumptions on the off-diagonal elements are changed to $\beta_i \leq 0$ for $1 \leq i \leq n-1$ and $\gamma_i \geq 0$ for $2 \leq i \leq n$?