Graded Homework 1 Applied Linear Algebra 2 Spring 2024

The solutions are due on Canvas by 11:59 PM on Friday, January 26, 2024

Problem 1.1

Suppose $A \in \mathbb{C}^{m\times n}$. Consider the matrix norm $\|A\|$ induced by the two vector 1-norms $\|x\|_1$ and $\|y\|_1$ for $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ respectively,

$$\|A\| = \max_{\|x\|_1=1} \|Ax\|_1.$$

Is this induced norm the same as the matrix 1-norm defined by

$$\|A\|_1 = \max_{1\le i\le n} \|Ae_i\|_1?$$

If so prove it. If not give counterexample to disprove it.

Problem 1.2

Consider an $n \times n$ nonsingular upper triangular matrix where all nonzero elements in row i are equal to α_i, e.g., for $n = 4$

$$U_4 = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ & \alpha_2 & \alpha_2 & \alpha_2 \\ & & \alpha_3 & \alpha_3 \\ & & & \alpha_4 \end{pmatrix}$$

Assume that $\alpha_i \neq 0$ and show that the system $U_n x = b$ can be solved in significantly fewer than $n^2 + O(n)$ computations. Give your complexity result in the form $Cn^k + O(n^{k-1})$ where C is a constant independent of n and $k > 0$.

Problem 1.3

Given the the scalars, $\gamma_0, \ldots, \gamma_n$ and $\beta_1, \ldots, \beta_n$, a first order linear recurrence that determines the values of the scalars $\alpha_0, \ldots, \alpha_n$ is defined as follows:

$$\begin{aligned} \alpha_0 &= \gamma_0 \\ \alpha_i &= \beta_i \alpha_{i-1} + \gamma_i, \quad i = 1, \cdots, n. \end{aligned}$$

1.3.a Show that this recurrence code solves a linear system of equations to get the values of $\alpha_0, \ldots, \alpha_n$.

1.3.b Comment on any structural properties of the matrix and how they are exploited in the algorithm.

1.3.c How many operations are required to solve the system using the algorithm you described?

1.3.d How much storage is required for the algorithm you described?

Problem 1.4

If for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ two vector norms satisfy

$$|y^T x| \leq \|x\|_\alpha \|y\|_\beta$$

with equality attainable then they are called dual norms. For example, the 2-norm is its own dual due to the Cauchy-Schwarz inequality.

Show that the 1-norm and the ∞-norm are dual norms, i.e., show that

$$|y^T x| \leq \|x\|_1 \|y\|_\infty.$$

Problem 1.5

Recall that a matrix norm, $\|A\|$, can be induced from a vector norm $\|x\|_\alpha$ by the definition

$$\|A\| = \max_{\|x\|_\alpha = 1} \|Ax\|_\alpha.$$

(Such a matrix norm is called the natural norm in some literature, e.g., Isaacson and Keller, Analysis of Numerical Methods.

Let $\mathcal{S}(A)$ be the set of of all eigenvalues of a square matrix $A \in \mathbb{C}^{n \times n}$. The spectral radius, $\rho(A)$, is the value

$$\rho(A) = \max_{\lambda \in \mathcal{S}(A)} |\lambda|$$

is called the spectral radius of A. (This is due to the fact that it is the radius of the smallest circle in the complex plane that contains all of the eigenvalues of A.)

Show that for any induced matrix norm

$$\rho(A) \leq \|A\|.$$

Problem 1.6

Let $M \in \mathbb{R}^{n\times n}$. Recall, the definitions of singular and nonsingular square matrices:

- M is nonsingular if its columns (rows) are linearly independent.
- M is nonsingular if its null space, $\mathcal{N}(M)$, contains only the vector $0 \in \mathbb{R}^n$.
- M is singular if its columns (rows) are linearly dependent.
- M is singular if its null space, $\mathcal{N}(M)$, contains a vector $x \in \mathbb{R}^n$ with $x \neq 0$.

1.6.a. Show that M is singular if and only if $\lambda = 0$ is an eigenvalue of M.

1.6.b. Show that M is nonsingular if and only if there exists a unique matrix $M^{-1} \in \mathbb{R}^{n\times n}$ such that

$$MM^{-1} = M^{-1}M = I_n$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix.