ON THE UNIQUENESS OF INVARIANT MEASURES FOR THE STOCHASTIC INFINITE DARCY-PRANDTL NUMBER MODEL

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The infinite Darcy-Prandtl number model is an effective reduced model for describing convection in a fluid-saturated porous medium. It is well known that the deterministic model does not possess a unique invariant measure. In this work we study the dynamics of the infinite Darcy-Prandtl number model, under an additive stochastic forcing of its low modes. This is the so-called stochastic infinite Darcy-Prandtl number model. We prove that the stochastically forced system does indeed possess a unique invariant measure.

Keywords: stochastic infinite Darcy-Prandtl number model, convection in porous media, invariant measures, stochastic partial differential equations

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1. Introduction

The phenomenon of convection in a fluid saturated porous medium has been studied quite extensively, dating back to the work of Horton and Rogers [HR45] and Lapwood [L48]. An effective model for convection in a porous medium is the Darcy-Boussinesq system [N99]. Under a large Darcy-Prandtl number assumption this system can be reduced to the infinite Darcy-Prandtl model [DC98]. Both of these models have generated considerable interest lately. They find diverse applications in the fields of geothermal engineering, construction of thermal insulators, nuclear waste management, thermal enhanced oil recovery and hydraulic fracturing. Enhanced oil recovery now represents a 31 billion dollar market opportunity, in the United States alone [S09]. Furthermore, to meet the growing demand for energy and to slow down greenhouse gas emissions, nuclear power can be an important source. With the use of nuclear power comes the issue of disposal of radioactive waste, possibly via shallow land burial. All of the above processes involve modelling fluid flows in porous media [$S01$].

In [$P10$], [$PT09$], we show convergence of the invariant measures of the Darcy-Boussinesq system to those of the infinite Darcy-Prandtl number model, as the Darcy-Prandtl number approaches infinity. We would now like to consider the question of uniqueness of the invariant measures for the infinite Darcy-Prandtl model. For the deterministic system there is no uniqueness, as seen by constructing measures consisting of point masses centered at various steady states [$WS09$]. These multiple steady states then will give rise to multiple invariant measures.

One possible approach is to consider the deterministic system forced by a white noise. Heuristically, the noise will connect the various branches of the attractor, which would not be able to interact without the presence of the noise. Essentially, the stochastic forcing introduces a probability of being "kicked" from one steady state to another, an impossibility without the noise. The noise will thus connect all disconnected branches of the attractor to yield an invariant measure. We pay the price however by having to work in a probabilistic setting on a Banach space.

Techniques of adding noise to deterministic partial differential equations have become quite popular. There is a vast literature on these so-called stochastic partial differential equations. See [$DZ96$] for a detailed treatment of such techniques. The breadth of techniques covered in [$DZ96$] are of a very technical nature. The authors treat the case of infinite-dimensional noise. For our case however, a finite-dimensional noise will suffice. Although there is no general theory to treat equations forced by degenerate or finite-dimensional noise, there are various results that provide us with valuable insight into the existence and uniqueness of invariant measures for dissipative stochastic systems. We recall certain prominent results relevant in our setting.

Mattingly, in his PhD dissertation, considered the 2d stochastic Navier-Stokes equations forced by degenerate white noise [$M98$]. He proved uniqueness of the invariant measure for this equation. Later, along with E and Sinai he studied this problem further, deriving various results that were reported in [$MES01$]. The methodology developed by them was extended to a host of other stochastic partial differential equations such as the stochastic Ginzburg-Landau and stochastic Cahn-Hilliard. These equations were considered by E and Liu [$E02$]. Recently these techniques were also applied to the 3d truncated stochastic Boussinesq system by Wu and Lee [$Wu04$].

Inspired by the above results we propose to consider the infinite Darcy-Prandtl number model in a stochastic setting. Essentially we add a finite-dimensional white noise term to the infinite Darcy-Prandtl number model to yield a stochastic partial differential equation. This enables us to borrow a variety of techniques from the literature on dissipative stochastic partial differential equations forced via finite-dimensional noise.

Our goal in the current manuscript is to apply results directly from [$E02$] to the stochastic infinite Darcy-Prandtl number model. E and Liu in [$E02$] prove existence of a unique invariant measure for a general dissipative stochastic PDE, under certain suitable conditions. We will systematically check that the stochastic infinite Darcy-Prandtl number model satisfies each one of these. We have organized our manuscript as follows. In section 2 we describe the mathematical formulation of the problem. Section 3 is aimed at validating the first two conditions as posed in [E02]: see the appendix. In section 4 we make various probabilistic estimates aimed at validating the remaining conditions. The results of these sections are brought together in section 5, where we state our main result via Theorem 5.1. We then offer some concluding remarks in section 6. For the benefit of the reader the appendix in section 7 recapitulates the essence of the results that we are applying from [E02] to our current work. In all estimates made henceforth C is a generic constant that can change in its value from line to line, and sometimes within the same line, if so required.

2. The Mathematical Formulation

2.1. *The infinite Darcy-Prandtl number model*

The physical space for the problem consists of a fluid-saturated porous medium, confined between two plates a distance of h units apart vertically. The porous layer is of length L_x in the x-direction and length L_y in the y-direction. The bottom plate is heated to a temperature T_2 and the top plate is cooled to a temperature T_1 where $T_2 > T_1$. In order to non-dimensionalize the problem we measure length in units of the layer thickness h, and time in units of the thermal diffusion time scale $\frac{h^2}{\kappa}$, where κ is the coefficient of thermal diffusion. Thus the fluid occupies the non-dimensional region

$$X = [0, \Lambda_x] \times [0, \Lambda_y] \times [0, 1]. \tag{2.1}$$

Here $\Lambda_x = \frac{L_x}{h}$ and $\Lambda_y = \frac{L_y}{h}$.

The differential heating induces convection in the fluid, which is modeled by a set of coupled partial differential equations—see [N99], [DC98]—known as the Darcy-Boussinesq system.

A key parameter in the above system is the Darcy-Prandtl number. This is defined as

$$Pr_D = \frac{\nu h^2}{\kappa K}. \tag{2.2}$$

Here, K is the Darcy permeability coefficient, and ν is the kinematic fluid viscosity. The Darcy-Prandtl number essentially measures the ability of the porous medium to transport fluid. A typical range of values for K is 10^{-8}–10^{-16}. When the porous medium in question has very low permeability, say $O(10^{-16})$ (such as very tightly packed sand, clay, or granite), the result is a very large Darcy-Prandtl number.

The infinite Darcy-Prandtl number model is obtained via formally taking the limit as Pr_D approaches ∞ in the Darcy-Boussinesq system [DC98]. The model is described by the following set of coupled partial differential equations along with a free-slip set of boundary conditions:

$$\mathbf{u} + \nabla p = Ra_D\ \mathbf{k}T, \tag{2.3}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T, \tag{2.4}$$

$$u_3|_{z=0,1} = 0,\ \left.\frac{\partial u_1}{\partial z}\right|_{z=0,1} = \left.\frac{\partial u_2}{\partial z}\right|_{z=0,1} = 0, \tag{2.5}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.6}$$

$$T|_{z=0} = 1, T|_{z=1} = 0, \mathbf{u}|_{t=0} = \mathbf{u}_0, T|_{t=0} = T_0. \tag{2.7}$$

On the side walls, periodic boundary conditions are imposed for convenience.

Here $\mathbf{u}$ is the seepage velocity, T is the temperature field, and $\mathbf{k}$ is the upward pointing unit vector. The parameter in the system is the Darcy-Rayleigh number defined as

$$Ra_D = \frac{g\alpha(T_2 - T_1)Kh}{\nu\kappa}, \tag{2.8}$$

where g is the gravitational acceleration.

Note the system is not equipped with homogeneous boundary conditions. This is circumvented by introducing a change of variable

$$T = \theta + \gamma(z). \tag{2.9}$$

Here $\gamma(z)$ is a background temperature profile: see [$CD96$]. We require that

$$\gamma(1) = 0,\ \gamma(0) = 1. \tag{2.10}$$

Thus θ will satisfy homogenous boundary conditions,

$$\theta|_{z=0,1} = 0. \tag{2.11}$$

Inserting the above into (2.3)–(2.4) yields

$$\nabla p + \mathbf{u} = Ra_D\ \mathbf{k}\theta, \tag{2.12}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.13}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta + u_3\gamma'(z) = \Delta\theta + \gamma''(z), \tag{2.14}$$

$$\theta|_{z=0,1} = 0. \tag{2.15}$$

We will suppose $\gamma'(z)$ has the form

$$\gamma'(z) = 0,\ 0 \le z \le 1 - \delta, \tag{2.16}$$

$$\gamma'(z) = \frac{1}{\delta},\ 1 - \delta < z \leq 1, \tag{2.17}$$

so that

$$\gamma''(z) = 0. \tag{2.18}$$

Appropriate choices of δ will be made later, as required.

2.2. *The stochastic infinite Darcy-Prandtl number model*

Notice from (2.12), the velocity is completely determined by the temperature field modulo the pressure. We will use this to our advantage and convert (2.12)–(2.14) to a single equation. We apply the Leray-Hopf projector, denoted P, ([T97]), to (2.12) and insert the result into (2.14) to yield the following equation

$$\frac{\partial \theta}{\partial t} = \Delta\theta - Ra_D P(\mathbf{k}\theta) \cdot \nabla\theta - Ra_D \gamma'(z)\, P(\theta), \tag{2.19}$$

$$\theta|_{z=0,1} = 0. \tag{2.20}$$

An appropriate white noise is added to (2.19) to yield the stochastic infinite Darcy-Prandtl number model

$$\mathrm{d}\theta = (\Delta\theta - Ra_D P(\mathbf{k}\theta) \cdot \nabla\theta - Ra_D \gamma'(z)\, P(\theta))\, \mathrm{d}t + \mathrm{d}\mathbb{W}, \tag{2.21}$$

$$\theta|_{z=0,1} = 0. \tag{2.22}$$

We now choose $\delta = 2CRa_D$—the reason for this will become clear in Lemma 3.1—where C is the Poincaré constant that arises in the Poincaré inequality

$$|\theta(t)|_2^2 \leq C|\nabla\theta(t)|_2^2. \tag{2.23}$$

We pause and ask the following question. Does there exist a unique invariant measure for (2.21)? We answer this question in the affirmative by applying results from [E02].

2.3. *Stochastic preliminaries*

In this section we discuss the stochastic framework on which much of the subsequent analysis relies. The key idea is to perform stochastic analysis on a Banach space. To this end we would like to discuss the structure of the noise and spaces used. For details the reader is referred to [DZ96]. We follow the presentation of [E02].

We consider (2.21) yet again

$$\mathrm{d}\theta = (\Delta\theta - Ra_D P(\mathbf{k}\theta) \cdot \nabla\theta - Ra_D \gamma'(z)\ P(\theta))\, \mathrm{d}t + \mathrm{d}\mathbb{W},\ t \geq 0,\ \theta(0) = \theta_0. \tag{2.24}$$

The noise $\mathbb{W}(\cdot, t)$ is of the form

$$\mathbb{W}(x, \omega, t) = \sum \sigma_k \omega_k(\omega, t) e_k(x), \tag{2.25}$$

where the ω_k's are independent standard brownian motions which generate the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The $\sigma_k \in \mathbb{R}$ are coordinate-wise standard deviations, and the $e_k(x)$ form a basis for $L^2(X)$. Expectations are taken with respect to $\mathbb{P}$. The phase space for us is $H = L^2(X)$. This is equipped with the standard inner product $\langle \cdot, \cdot \rangle$. Defining noise with this structure facilitates taking inner products. This becomes important in the energy estimates performed henceforth. We will assume (2.21) can be solved for almost all $\omega \in \Omega$ and defines a continuous markovian semigroup, denoted

$$\phi^{\omega}_{s,t}\theta_0 = \theta(s, t; \omega, \theta_0). \tag{2.26}$$

A probability measure μ on the phase space H equipped with the Borel σ-algebra is invariant if and only if

$$\int_H F(\theta)\mu(\mathrm{d}\theta) = \int_H \mathbb{E}[F(\phi^{\omega}_t \theta)]\mu(\mathrm{d}\theta), \tag{2.27}$$

for all continuous bounded functions F on H and all $t \geq 0$.

An invariant measure μ can be extended to a measure μ_p on the path space $C((-\infty, 0], H)$. First, we define a cylinder set

$$A = \{\theta(s) \in C((-\infty, 0], H), \theta(t_i) \in A_i, i = 0, ...n\}, \tag{2.28}$$

where $t_0 < t_1 < t_2 < ...< t_n < 0$, and the A_i are Borel subsets of H. Define $B \subset H \times \Omega$ by

$$B = \left\{(\theta, w), \theta \in A_0, \phi_{t_0,t_i} \in A_i, i = 1, 2, ..n\right\}, \tag{2.29}$$

and define $\mu_p(A) = (\mu \times \mathbb{P})(A)$. Then μ_p is consistent on cylinder sets and can be extended to the natural σ-algebra by Kolmogorov's extension theorem.

We also briefly recount the concept of gibbsian dynamics of the low modes. We partition the phase space $H = L^2(X)$ into a space of high modes and low modes:

$$H_l = \text{span}\{e_k(x), k \leq N\}, \ H_h = \text{span}\{e_k(x), k > N\}, \tag{2.30}$$

where $H = H_l \oplus H_h$. We denote by P_l the projection operator from H to H_l and by P_h the projection operator from H to H_h. Thus a solution to (2.21) is written as $\theta(t) = (l(t), h(t))$ where $P_l(\theta(t)) = l(t)$ and $P_h(\theta(t)) = h(t)$. We can thus rewrite (2.21) in terms of its high mode and low mode components as

$$\mathrm{d}l(t) = \Delta l(t)\,\mathrm{d}t - P_l(Ra_D P(\mathbf{k}\theta) \cdot \nabla\theta - Ra_D \gamma^{'}(z)\, P(\theta))\,\mathrm{d}t + \mathrm{d}\mathbb{W}(t), \tag{2.31}$$

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} = \Delta h(t) - P_h(Ra_D P(\mathbf{k}\theta) \cdot \nabla\theta - Ra_D \gamma^{'}(z)\, P(\theta)). \tag{2.32}$$

A number of conditions are imposed on (2.21): see the appendix. Note that given an ergodic invariant measure μ, for μ_p-almost all $\theta(\cdot) \in C((-\infty, 0], H)$ we have that

$$\lim_{t_0 \to -\infty} \frac{1}{t - t_0} \int_{t_0}^{t} K(\theta(s))\,\mathrm{d}s \leq \beta, \tag{2.33}$$

for some positive function K (see condition 2 in the appendix).

We next define the set $U \subseteq C((-\infty,0],H)$ to consist of all $v:(-\infty,0]\to H$ such that v satisfies the estimates derived in Lemma 4.4 and the integral estimate derived above. Then due to conditions 1 and 2, which we prove henceforth, and the ergodicity assumption, we have that $\mu_p(U)=1$. Also we will use $l(t)$ to denote the value of the low modes of the solution at time t, and L^t to denote the entire trajectory of the low modes from $-\infty$ to t. Therefore we have that $l(t)\in H_l$ and $L^t\in C((-\infty,t],H_l)$, and that $l(s)=L^t(s)$ for $0\le s\le t$. We define a map $\Phi_s(L^t,h_0)$ which is a solution to (2.32) at time s with initial condition h_0 and low mode forcing L^t.

Note that $\Phi_s(L^t,h_0)$ only depends on the information of L^t between 0 and s. Therefore we can define $\Phi_{t_0,s}(L^t,h_0)$ for solutions starting from t_0 rather than time 0. We will suppose N is large enough so that the requisite condition from Lemma 5.1 holds. Then we can solve for the future of l using the gibbsian dynamics

$$\mathrm{d}l(t)=[\Delta l(t)+G(l(t),\Phi_t(L^t))]\,\mathrm{d}t=\mathrm{d}\mathbb{W}(t). \tag{2.34}$$

Here

$$G(l,h)=P_l(Ra_D P(\mathbf{k}(l+h))\cdot\nabla(l+h)+Ra_D P(l+h))). \tag{2.35}$$

Thus we have a closed form for the dynamics of the low modes given an initial past. The following difference operator also appears often:

$$D(f,g_1,g_2)=G(f,g_1)-G(f,g_2). \tag{2.36}$$

Also we will abbreviate the nonlinear terms in (2.21) as

$$R(\theta)=-Ra_D\ P(\mathbf{k}\theta)\cdot\nabla\theta-Ra_D\ \gamma'(z)P(\theta). \tag{2.37}$$

3. Proof of first two conditions

We now proceed systematically to verify the conditions set forth by E and Liu in [$E02$]: again, see the appendix. Essentially we want to follow the ideas in [$E02$] and reduce the infinite dimensional dynamics of the stochastic infinite Darcy-Prandtl number model to the finite-dimensional gibbsian dynamics. This will facilitate the use of Girsanov's theorem to yield a unique invariant measure. In this section we state and prove two lemmas from which the first two conditions in [$E02$] will be a direct consequence.

Lemma 3.1. *For the infinite Darcy-Prandtl number model there exist constants $\eta>0$ and $k_0\ge 0$ such that*

$$\langle\Delta\theta,\theta\rangle_2+\langle -Ra_D\ P(\mathbf{k}\theta)\cdot\nabla\theta-Ra_D\ \gamma'(z)P(\theta),\theta\rangle_2\le-\eta|\theta(t)|_2^2+k_0. \tag{3.1}$$

Proof. From the form of (2.19), and the assumed choice of δ we have that

$$\begin{aligned}\langle\Delta\theta,\theta\rangle_2+\langle R(\theta),\theta\rangle_2 &= \langle\Delta\theta,\theta\rangle_2+\left\langle -Ra_D\ P(\mathbf{k}\theta)\cdot\nabla\theta - Ra_D\frac{1}{2CRa_D}\ P(\theta),\theta\right\rangle_2\\ &\le -|\nabla\theta|_2^2+\frac{1}{2C}|\theta|_2^2\\ &\le -\frac{1}{C}|\theta|_2^2+\frac{1}{2C}|\theta|_2^2\\ &\le -\frac{1}{2C}|\theta(t)|_2^2.\end{aligned}$$

This follows via integration by parts, Poincaré's inequality

$$|\theta|_2^2 \le C|\nabla\theta|_2^2, \tag{3.2}$$

and the estimate on the Leray projector

$$|P(\theta)|_2^2 \le |\theta|_2^2. \tag{3.3}$$

Thus the lemma is proved for, say, $\eta=\frac{1}{2C}$ and $k_0=0$. □

Thus condition 1 (see appendix) from [$E02$] is satisfied.

We now look at the difference of the non-linear terms, $R(\theta_1-\theta_2)$, where θ_1 and θ_2 are two different solutions to the infinite Darcy-Prandtl number model such that $\theta_1=l+h_1$ and $\theta_2=l+h_2$. We set $\rho=\theta_1-\theta_2=h_1-h_2$ and state the lemma:

Lemma 3.2. *For the infinite Darcy-Prandtl number model there exists a constant $\alpha\in[0,1)$ and a non negative function $K(\theta_1)$ such that*

$$\langle R(\rho),\rho\rangle \le \alpha\,\langle -\Delta\rho,\rho\rangle + K(\theta_1)|\rho|_2^2. \tag{3.4}$$

Furthermore there exists a uniform constant C such that

$$\int_H K(\theta_1)\,\mathrm{d}\mu(\theta_1) \le C. \tag{3.5}$$

Proof. Recall from the form of (2.21) we have

$$R(\rho) = -Ra_D\ P(\mathbf{k}\theta_1)\cdot\nabla\theta_1 + Ra_D\ P(\mathbf{k}\theta_2)\cdot\nabla\theta_2 + Ra_D\ \gamma'(z)\ P(\rho). \tag{3.6}$$

We will rewrite $R(\rho)$ as

$$R(\rho) = -Ra_D\ P(\mathbf{k}\theta_2)\cdot\nabla\rho - Ra_D\ \nabla\theta_1\cdot P(\mathbf{k}\rho) + Ra_D\ \gamma'(z)P(\rho). \tag{3.7}$$

Without loss of generality we assume $\theta_1\ge\theta_2$. We multiply (3.7) through by $\rho=$

$\theta_1 - \theta_2$ and integrate by parts to yield

$$\begin{aligned}\langle R(\rho),\rho\rangle &\le \langle -Ra_D\ P(\mathbf{k}\theta_2)\cdot\nabla\rho - Ra_D\ \nabla\theta_1\cdot P(\mathbf{k}\rho) + Ra_D\gamma'(z)P(\rho),\rho\rangle\\ &= -Ra_D\ \int_H (\theta_1)_z P(\rho)\rho\,\mathrm{d}\mathbf{x} + Ra_D\int_H \gamma'(z)P(\rho)\rho\,\mathrm{d}\mathbf{x}\\ &\le Ra_D|B(\rho,\theta_1,\rho)| + \frac{1}{2C}\int_H |\rho|^2\,\mathrm{d}\mathbf{x}\\ &\le \frac{1}{4}|\nabla\rho|_2^2 + C|\rho|_2|\nabla\rho|_2|\nabla\theta_1|_2 + C|\rho|_2^2\\ &\le \frac{1}{4}|\nabla\rho|_2^2 + C|\nabla\theta_1|_2^2||\rho|_2^2 + C|\rho|_2^2\\ &\le \frac{1}{4}\,\langle -\Delta\rho,\rho\rangle_2 + (C+|\nabla\theta_1|_2^2)|\rho|_2^2.\end{aligned}$$

This follows via estimates for the tri-linear term, the Cauchy with epsilon inequality, Poincaré's inequality and the choice of $\gamma'(z)$. Thus the lemma is proved for, say, $\alpha = \frac{1}{4}$ and $K(\theta_1) = (C + |\nabla\theta_1|_2^2)$. In order to see how we obtain

$$\int_H K(\theta_1)\,\mathrm{d}\mu(\theta_1) \le C, \tag{3.8}$$

see the proof for Lemma 4.2. □

4. Estimates on the growth rate of energy and enstrophy

Our goal in this section is to derive estimates for the energy $\mathbb{E}[|\theta|_2^2]$ and the enstrophy $\mathbb{E}[|\nabla\theta|_2^2]$. This is necessary to derive certain lemmas which are crucial for proving the next set of conditions.

4.1. *A priori estimates on the energy*

We begin with estimates on the energy. We apply Itô's lemma on the map

$$\theta(t) \mapsto \frac{1}{2}|\theta(t)|_2^2. \tag{4.1}$$

This yields

$$\begin{aligned}\tfrac{1}{2}\mathrm{d}|\theta(t)|_2^2 = [Ra_D \textstyle\int_\Omega \gamma'(z)P(\mathbf{k}\theta(t))\theta(t)\,\mathrm{d}\mathbf{x} -\ |\nabla\theta(t)|_2^2 - \langle \mathbf{u}\cdot\nabla\theta,\theta\rangle_2]\,\mathrm{d}t\\ -\langle\theta,\mathrm{d}\mathbb{W}\rangle_2 + \epsilon_0\,\mathrm{d}t.\end{aligned} \tag{4.2}$$

Here $\epsilon_0 = \sum|\sigma_k|^2$. Via the appropriate choice for $\gamma(z)$ we obtain

$$\frac{1}{2}\mathrm{d}|\theta(t)|_2^2 \le \left[\frac{1}{2C}|\theta(t)|_2^2 -\ |\nabla\theta(t)|_2^2\right]\mathrm{d}t + \epsilon_0\,\mathrm{d}t - \langle\theta,\mathrm{d}\mathbb{W}\rangle_2\,. \tag{4.3}$$

Define a stopping time T, for any given H, by

$$T = \inf\left\{t : |\theta(t)|_2^2 \ge H^2\right\}. \tag{4.4}$$

Integrate (4.3) from 0 to $t \wedge T$ and take expectations to yield

$$\begin{aligned}\tfrac{1}{2}\mathbb{E}[|\theta(t\wedge T)|_2^2] \leq \tfrac{1}{2}\mathbb{E}[|\theta(0)|_2^2] + \tfrac{1}{2C}\textstyle\int_0^{t\wedge T}\mathbb{E}[|\theta(s)|_2^2]\,\mathrm{d}s - \int_0^{t\wedge T}\mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}s \\ -\mathbb{E}[\textstyle\int_0^{t\wedge T}\langle\theta,\mathrm{d}\mathbb{W}\rangle_2] + \epsilon_0(t\wedge T).\end{aligned} \tag{4.5}$$

Now define the quantity

$$M_t^T = \int_0^t \langle \theta(s\wedge T), \mathrm{d}\mathbb{W}\rangle_{L^2}\,. \tag{4.6}$$

We can show that the quadratic variation of M_t^T is finite, implying that M_t^T is a martingale and so

$$\mathbb{E}[M_t^T] = 0. \tag{4.7}$$

We can therefore use the optional sampling theorem to conclude

$$E[M_{t\wedge T}^T] = 0. \tag{4.8}$$

Hence we can take $H \to \infty$ so $T \to \infty$ and thus $t \wedge T \to t$. Incorporating these limits in (4.5) yields

$$\frac{1}{2}\mathbb{E}[|\theta(t)|_2^2] + \int_0^t \mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}s \leq \frac{1}{2}\mathbb{E}[|\theta(0)|_2^2] + \frac{1}{2C}\int_0^t \mathbb{E}[|\theta(s)|_2^2]\,\mathrm{d}s + \epsilon_0 t. \tag{4.9}$$

Thus application of Poincaré's inequality yields

$$\frac{1}{2}\mathbb{E}[|\theta(t)|_2^2] + \frac{1}{C}\int_0^t \mathbb{E}[|\theta(s)|_2^2]\,\mathrm{d}s \leq \frac{1}{2}\mathbb{E}[|\theta(0)|_2^2] + \frac{1}{2C}\int_0^t \mathbb{E}[|\theta(s)|_2^2]\,\mathrm{d}s + \epsilon_0 t. \tag{4.10}$$

Thus we have

$$\frac{1}{2}\mathbb{E}[|\theta(t)|_2^2] + \frac{1}{2C}\int_0^t \mathbb{E}[|\theta(s)|_2^2]\,\mathrm{d}s \leq \frac{1}{2}\mathbb{E}[|\theta(0)|_2^2] + \epsilon_0 t, \tag{4.11}$$

and an application of Gronwall's lemma yields

$$\mathbb{E}[|\theta(t)|_2^2] \leq e^{-2C_2 t}\mathbb{E}[|\theta(0)|_2^2] + \frac{\epsilon_0}{C_2}(1 - e^{-2C_2 t}). \tag{4.12}$$

Here $C_2 = \frac{1}{2C}$.

We now state a lemma that enables us to derive a uniform bound on the L^2-norm of θ.

Lemma 4.1. *Consider a θ that is a solution to the stochastic infinite Darcy-Prandtl number model. For an invariant measure μ on H, there exists a constant C such that the following estimate holds uniformly:*

$$\int_H |\theta|_2^2\,\mathrm{d}\mu(\theta) \leq C. \tag{4.13}$$

Proof. For any $\epsilon > 0$, there is a b_ϵ such that $\mu\left\{\theta : |\theta|_2^2 \leq b_\epsilon\right\} \geq 1 - \epsilon$. We define the set

$$B_\epsilon = \left\{\theta : |\theta|_2^2 \leq b_\epsilon\right\}. \tag{4.14}$$

Thus we have that for any H and $t > 0$,

$$\begin{aligned}\int_H (|\theta|_2^2 \wedge H)\,\mathrm{d}\mu(\theta) &= \int_H \mathbb{E}(|\phi^{\omega}_{0,t}\theta|_2^2 \wedge H)\,\mathrm{d}\mu(\theta)\\ &\le \int_{B_\epsilon} \mathbb{E}(|\phi^{\omega}_{0,t}\theta|_2^2)\,\mathrm{d}\mu(\theta) + H\epsilon\\ &\le e^{-2C_2 t} b_\epsilon + \frac{\epsilon_0}{C_2}(1 - e^{-2C_2 t}).\end{aligned}$$

This follows via the estimates derived in (4.12). We now let $t \to \infty$ and $H \to \infty$ and obtain the result, as ϵ was arbitrary. This proves the lemma. □

4.2. *A priori estimates on the enstrophy*

We will now derive estimates for $\mathbb{E}[|\nabla\theta(t)|_2^2]$. We apply Itô's lemma on the map

$$\theta(t) \mapsto \frac{1}{2}|\nabla\theta(t)|_2^2. \tag{4.15}$$

This yields

$$\begin{aligned}\tfrac{1}{2}\mathrm{d}|\nabla\theta(t)|_2^2 = [-|\Delta\theta(t)|_2^2 + Ra_D \textstyle\int_\Omega \gamma'(z)\Delta\theta P(\theta)\,\mathrm{d}\mathbf{x}]\,\mathrm{d}t\\ - \langle\Delta\theta, \mathrm{d}\mathbb{W}\rangle_2 + (\langle\mathbf{u}\cdot\nabla\theta, \Delta\theta\rangle_2 + \epsilon_1)\,\mathrm{d}t.\end{aligned} \tag{4.16}$$

Here $\epsilon_1 = \sum k^2|\sigma_k|^2$. We now define the stopping time T, for any given H, by

$$T = \inf\left\{t : |\nabla\theta(t)|_2^2 \ge H^2\right\}. \tag{4.17}$$

We use the Cauchy-Schwartz, H¨older's, Poincaré's, and Young's inequalities and choose

$$\gamma'(z) = \frac{1}{8C_3}, \tag{4.18}$$

where C_3 is the constant that arises in the embedding of $H^2(X) \hookrightarrow H_0^1(X)$, i.e.,

$$|\nabla\theta|_2^2 \le C_3|\Delta\theta|_2^2, \tag{4.19}$$

to obtain

$$\begin{aligned}\tfrac{1}{2}\mathrm{d}|\nabla\theta(t)|_2^2 \le [-|\Delta\theta(t)|_2^2 + Ra_D\ |\theta|_\infty|\nabla\theta|_2|\Delta\theta|_2 + \tfrac{1}{8C_3}\ |\nabla\theta|_2^2]\,\mathrm{d}t\\ - \langle\Delta\theta, \mathrm{d}\mathbb{W}\rangle_2 + \epsilon_1\,\mathrm{d}t.\end{aligned} \tag{4.20}$$

Recall the interpolation inequality

$$|\nabla\theta|_2 \le C|\Delta\theta|_2^{\frac{1}{2}}|\theta|_2^{\frac{1}{2}}, \tag{4.21}$$

and use this to bound the nonlinear term:

$$\begin{aligned}-Ra_D \int_\Omega (P(k\theta)\cdot\nabla\theta)\Delta\theta\,\mathrm{d}\mathbf{x} &\le Ra_D\ ||\theta||_\infty \left(\int_\Omega |\nabla\theta|^2\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \left(\int_\Omega |\Delta\theta|^2\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}\\ &\le C\left(\int_\Omega |\theta|^2\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{4}} \left(\int_\Omega |\Delta\theta|^2\,\mathrm{d}\mathbf{x}\right)^{\frac{3}{4}}\\ &\le \frac{3}{4}\int_\Omega |\Delta\theta|^2\,\mathrm{d}\mathbf{x} + C\int_\Omega |\theta|^2\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\mathrm{d}|\nabla\theta|_2^2 &= 2\left[-|\Delta\theta|_2^2 + \frac{1}{8C_3}\ |\nabla\theta|_2^2]dt - Ra_D\int_\Omega (P(k\theta)\cdot\nabla\theta)\Delta\theta\,\mathrm{d}\mathbf{x}\right]\mathrm{d}t \\
&\quad - \langle\Delta\theta, \mathrm{d}\mathbb{W}\rangle_2 + \epsilon_1\,\mathrm{d}t \\
&\le 2\left[-|\Delta\theta|_2^2 + \frac{3}{4}|\Delta\theta|_2^2 + \frac{1}{8C_3}\ |\nabla\theta|_2^2 + C_3\int_\Omega |\theta|^2\,\mathrm{d}\mathbf{x}\right]\mathrm{d}t \\
&\quad - \langle\Delta\theta, \mathrm{d}\mathbb{W}\rangle_2 + \epsilon_1\,\mathrm{d}t.
\end{aligned}$$

Now integrating from 0 to $t\wedge T$ and taking expectation on both sides yields

$$\begin{aligned}
\tfrac{1}{2}\mathbb{E}|\nabla\theta(t\wedge T)|_2^2 &\le -\tfrac{1}{4}\int_0^{t\wedge T}\mathbb{E}|\Delta\theta(s)|_2^2\,\mathrm{d}s + \tfrac{1}{8C_3}\ \int_0^{t\wedge T}\mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}s \\
&+C\int_0^{t\wedge T}\mathbb{E}|\theta(s)|_2^2\,\mathrm{d}s - \int_0^{t\wedge T}\mathbb{E}\,\langle\Delta\theta, \mathrm{d}\mathbb{W}\rangle_2 + \epsilon_1(t\wedge T).
\end{aligned} \tag{4.22}$$

We now define the quantity

$$M_t^T = \int_0^t \langle\Delta\theta(s\wedge T), \mathrm{d}\mathbb{W}(s)\rangle_2\,, \tag{4.23}$$

and we can show that the quadratic variation of M_t^T is finite, implying that M_t^T is a martingale and therefore $\mathbb{E}[M_t^T] = 0$. Then the optional sampling theorem yields $\mathbb{E}[M_{t\wedge T}^T] = 0$. Hence we can take $H\to\infty$, so $T\to\infty$ and thus $t\wedge T\to t$. Again incorporating these limits in (4.22) gives

$$\begin{aligned}
\tfrac{1}{2}\mathbb{E}[|\nabla\theta(t)|_2^2] - \tfrac{1}{8C_3}\int_0^t\mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}s + \tfrac{1}{4}\int_0^t\mathbb{E}[|\Delta\theta(s)|_2^2]\,\mathrm{d}s \\
\le \tfrac{1}{2}\mathbb{E}[|\nabla\theta(0)|_2^2] + C\int_0^t\mathbb{E}[|\theta(s)|_2^2]\,\mathrm{d}s + \epsilon_1 t.
\end{aligned} \tag{4.24}$$

Note the embedding of $H^2(X)\hookrightarrow H^1(X)$:

$$|\nabla\theta|_2^2 \le C_3|\Delta\theta|_2^2. \tag{4.25}$$

Using the above in conjunction with the Poincaré's inequality yields

$$\begin{aligned}
\tfrac{1}{2}\mathbb{E}|\nabla\theta(t)|_2^2 - \tfrac{1}{8C_3}\int_0^t\mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}s + \tfrac{1}{4C_3}\int_0^t\mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}s \\
\le \tfrac{1}{2}\mathbb{E}[|\nabla\theta(0)|_2^2] + C\int_0^t\mathbb{E}[|\theta(s)|_2^2]\,\mathrm{d}s + \epsilon_1 t,
\end{aligned} \tag{4.26}$$

and so

$$\frac{1}{2}\mathbb{E}[|\nabla\theta(t)|_2^2] + C_4\int_0^t\mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}s \le \frac{1}{2}\mathbb{E}[|\nabla\theta(0)|_2^2] + (C_5+\epsilon_1)t. \tag{4.27}$$

Here $C_4 = \frac{1}{8C_3}$ and C_5 is the uniform estimate on $\mathbb{E}[|\theta(s)|_2^2]$ via (4.12). Hence an application of Gronwall's lemma yields

$$\mathbb{E}[|\nabla\theta(t)|_2^2] \le e^{-2C_4 t}(\mathbb{E}[|\nabla\theta(0)|_2^2]) + \left(\frac{\epsilon_1+C_5}{2C_4}\right)(1-e^{-2C_4 t}). \tag{4.28}$$

We now state a lemma that enables us to derive a uniform bound on the L^2-norm of $\nabla\theta$.

Lemma 4.2. *Consider a θ that is a solution to the stochastic infinite Darcy-Prandtl number model. For an invariant measure μ on H, there exists a constant C such that if $|\nabla\theta(0)|_2^2 \le C$, then the following estimate holds uniformly:*

$$\int_H |\nabla\theta|_2^2 \,\mathrm{d}\mu(\theta) \le C. \tag{4.29}$$

Proof. The proof could follow via mimicking the methods for the L^2-norm of θ. However, we provide an alternate proof. It follows from the a priori estimates on $\mathbb{E}[|\nabla\theta|_2^2]$ that

$$\frac{1}{2}\mathbb{E}[|\theta(t)|_2^2] + \int_{t_0}^t \mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}s \le \frac{1}{2}\mathbb{E}[|\theta(t_0)|_2^2] + \frac{1}{2C}\int_{t_0}^t \mathbb{E}[|\theta(s)|_2^2]\,\mathrm{d}s + \epsilon_0(t-t_0). \tag{4.30}$$

This implies that

$$\begin{aligned}\lim_{t_0\to\infty}\frac{1}{t-t_0}\int_{t_0}^t \mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}s &\le \lim_{t_0\to\infty}\frac{1}{t-t_0}\frac{1}{2}\mathbb{E}[|\theta(t_0)|_2^2] \\ &\quad + \lim_{t_0\to\infty}\frac{1}{t-t_0}\frac{1}{2C}\int_{t_0}^t \mathbb{E}[|\theta(s)|_2^2]\,\mathrm{d}s + \epsilon_0 \\ &\le C,\end{aligned}$$

using the uniform estimates on $\mathbb{E}[|\theta(s)|_2^2]$. Thus, from ergodicity, it follows that

$$\lim_{t_0\to\infty}\frac{1}{t-t_0}\int_{t_0}^t \mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}s = \int_H \mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}\mu(\theta) \le C. \tag{4.31}$$

However an application of Fubini's theorem for nonnegative integrands implies that

$$\int_H \mathbb{E}[|\nabla\theta(s)|_2^2]\,\mathrm{d}\mu(\theta) = \mathbb{E}\int_H [|\nabla\theta(s)|_2^2]\,\mathrm{d}\mu(\theta) \le C, \tag{4.32}$$

and from the definition of invariant measure we obtain

$$\mathbb{E}\int_H |\nabla\theta(s)|_2^2\,\mathrm{d}\mu(\theta) = \int_H |\nabla\theta(s)|_2^2\,\mathrm{d}\mu(\theta) \le C. \tag{4.33}$$

This proves the lemma. □

4.3. *Probabilistic estimate on growth rate of energy and enstrophy*

The following lemma gives a probabilistic estimate of the growth rate of $|\theta|_2^2$ and $|\nabla\theta|_2^2$.

Lemma 4.3. *For all $a \in (0,1)$, $\delta > 1$, there exists a K_1 such that if $|\theta_0|_2^2 + |\nabla\theta_0|_2^2 \le C_0$, then, for any $t \ge 0$,*

$$\mathbb{P}\left\{\begin{array}{c}|\theta(t)|_2^2 + |\nabla\theta(t)|_2^2 + C(\int_0^t |\nabla\theta(s)|_2^2\,\mathrm{d}s + \int_0^t |\Delta\theta(s)|_2^2\,\mathrm{d}s) \\ \le C_1 + C_2 t + K_1(t+1)^\delta\end{array}\right\} \ge 1-a. \tag{4.34}$$

Proof. From (4.3), (4.16) we have that

$$
\begin{aligned}
&|\theta(t)|_2^2 + |\nabla\theta(t)|_2^2 + C\left(\int_0^t |\nabla\theta(s)|_2^2\,\mathrm{d}s + \int_0^t |\Delta\theta(s)|_2^2\,\mathrm{d}s\right)\\
&\le C(|\theta(0)|_2^2 + |\nabla\theta(0)|_2^2) + (\epsilon_0 + C_5 + \epsilon_1)t + \int_0^t \langle\theta, \mathrm{d}\mathbb{W}\rangle_2 - \int_0^t \langle\Delta\theta, \mathrm{d}\mathbb{W}\rangle_2\\
&\le C_1 + C_2 t + \int_0^t \langle\theta, \mathrm{d}\mathbb{W}\rangle_2 - \int_0^t \langle\Delta\theta, \mathrm{d}\mathbb{W}\rangle_2\,.
\end{aligned}
$$

Consider the processes

$$M_t = \int_0^t \langle\theta, \mathrm{d}\mathbb{W}\rangle_2 \tag{4.35}$$

and

$$M_t^1 = -\int_0^t \langle\Delta\theta, \mathrm{d}\mathbb{W}\rangle_2\,. \tag{4.36}$$

For our purpose it suffices to show that, for $t \ge 0$,

$$\mathbb{P}\left\{M_t + M_t^1 \le \frac{K_1}{2}(t+1)^\delta\right\} \ge 1 - a. \tag{4.37}$$

We consider the quadratic variations of the processes M_t and M_t^1 to obtain

$$[M, M]_t \le (\sigma^*_{\max})^2 \int_0^t |\theta(s)|_2^2\,\mathrm{d}s \tag{4.38}$$

and

$$[M^1, M^1]_t \le (\Delta\sigma^*_{\max})^2 \int_0^t |\theta(s)|_2^2\,\mathrm{d}s, \tag{4.39}$$

where

$$(\sigma^*_{\max})^2 = \sup|\sigma_k|^2 \tag{4.40}$$

and

$$(\Delta\sigma^*_{\max})^2 = \sup|k^2\sigma_k|^2. \tag{4.41}$$

Note we have obtained estimates on the L^2-norm of θ. We thus proceed by defining the following events

$$B_k = \left\{\sup_{s\in[0,k]} |M_s| \ge \frac{K_1}{4}(k+1)^\delta\right\}. \tag{4.42}$$

We proceed by making an estimate of the probability of this event.

$$\begin{aligned}
\mathbb{P}\{B_k\} &\leq \frac{4^2}{(K_1)^2(k+1)^{2\delta}}\mathbb{E}[|M_s|^2] \quad \text{(Doob's inequality applied to } B_k\text{)} \\
&\leq \frac{4^2}{(K_1)^2(k+1)^{2\delta}}\mathbb{E}\left[\sup_{s\in[0,k]}|M_s|^2\right] \\
&\leq \frac{4^2}{(K_1)^2(k+1)^{2\delta}}\mathbb{E}[([M,M]_k)] \quad \text{(Burkholder-Davis-Gundy inequality)} \\
&\leq \frac{4^2}{(K_1)^2(k+1)^{2\delta}}\mathbb{E}\left[(\sigma^*_{\max})^2\int_0^t|\theta(s)|_2^2\right]\mathrm{d}s \quad \text{(estimate on } \mathbb{E}[([M,M|]_k)]\text{)} \\
&\leq \frac{4^2}{(K_1)^2(k+1)^{2\delta}}(\sigma^*_{\max})^2\int_0^k\mathbb{E}[|\theta(s)|_2^2]\,\mathrm{d}s \\
&\leq \frac{4^2}{(K_1)^2(k+1)^{2\delta}}(\sigma^*_{\max})^2Ck \quad \text{(estimate on } \mathbb{E}[|\theta(s)|_2^2]\text{)}.
\end{aligned}$$

Therefore we have that

$$\mathbb{P}\{B_k\} \leq \frac{Ck}{(k+1)^{2\delta}}. \tag{4.43}$$

We note that

$$\mathbb{P}\left\{M_t \leq \frac{K_1}{4}(t)^\delta\right\} = 1 - \mathbb{P}\left\{\bigcup_k B_k\right\} \geq 1 - \sum_k \mathbb{P}\{B_k\}. \tag{4.44}$$

We see that these sums are finite for sufficiently large δ. In particular $\delta > 1$ suffices. We note

$$\mathbb{P}\left\{M_t \leq \frac{K_1}{4}(t)^\delta\right\} \geq 1 - \sum_k \mathbb{P}\{B_k\}. \tag{4.45}$$

We can make the sum $\sum_k \mathbb{P}\{B_k\}$ arbitrarily small by increasing K_1, since

$$\sum_k \mathbb{P}\{B_k\} \leq \frac{1}{K_1^2}\sum_k \frac{Ck}{(k+1)^{2\delta}} \leq \frac{1}{K_1^2}\sum_k \frac{1}{k^p} \leq \frac{C}{K_1^2}. \tag{4.46}$$

Here $p > 1$. Thus, given $a \in (0,1)$, if we choose $K_1 = \sqrt{\frac{2C}{a}}$, we obtain

$$\mathbb{P}\left\{M_t > \frac{K_1}{4}t^\delta\right\} \leq \frac{a}{2}. \tag{4.47}$$

The same argument applies to the process M_t^1. Similar estimates can be made on $\mathbb{P}\{A_k\}$ where

$$A_k = \left\{\sup_{s\in[0,k]}|M_s^1| \geq \frac{K_1}{4}(k+1)^\delta\right\}. \tag{4.48}$$

Thus after performing a similar analysis as above, we can choose K_1 large enough to have

$$\mathbb{P}\left\{M_t^1 > \frac{K_1}{4}(t)^\delta\right\} \leq \frac{a}{2}, \tag{4.49}$$

and then combining (4.47) and (4.49) yields

$$\mathbb{P}\left\{M_t + M_t^1 \leq \frac{K_1}{2}(t+1)^\delta\right\} \geq 1-a, \tag{4.50}$$

for $t \geq 0$, as was required. This completes the proof of the lemma. □

We next prove the following lemma.

Lemma 4.4. *Let μ_p be a measure induced on $C((-\infty,0],L^2(X))$ by any given stationary measure μ. Fix any $K_0 \geq 0$ and $\delta > \frac{1}{2}$. Then for μ_p-almost every trajectory $\theta(s)$ in $C((-\infty,0],L^2(X))$, there exists a constant T_1 such that for $s \leq 0$ we have that*

$$|\theta(s)|_2^2 \leq \epsilon_0 + K_0 \min(T_1,|s|)^\delta. \tag{4.51}$$

Proof. The basic energy estimates give us

$$|\theta(t)|_2^2 \leq |\theta_0|_2^2 - C_1\int_{t_0}^t |\theta(s)|_2^2\,\mathrm{d}s + \epsilon_0(t-t_0) + \int_{t_0}^t \langle\theta(s),\mathrm{d}\mathbb{W}(s)\rangle_2\,. \tag{4.52}$$

Define the quantity

$$F_k(s) = -C_1\int_{-k}^s |\theta(t)|_2^2\,\mathrm{d}t + \int_{-k}^s \langle\theta(t),\mathrm{d}\mathbb{W}(t)\rangle_2\,. \tag{4.53}$$

By the above definition we have that, for any $k \geq 1$,

$$\sup_{s\in[-k,-k+1]} |\theta(s)|_2^2 \leq |\theta(-k)|_2^2 + \epsilon_0 + \sup_{s\in[-k,-k+1]} F_k(s). \tag{4.54}$$

We now define the event

$$A_k = \left\{\theta(s) : \sup_{s\in[-k,-k+1]} |\theta(t)|_2^2 \leq \epsilon_0 + K_0|k-1|^\delta\right\}. \tag{4.55}$$

Let

$$U_T = \bigcap_{k\geq T} A_k. \tag{4.56}$$

Then

$$U_T^c = \left(\bigcap_{k\geq T} A_k\right)^c = \bigcup_{k\geq T} A_k^c, \tag{4.57}$$

and we have that

$$\mu_p(U_T^c) = \mu_p\left(\bigcup_{k\geq T} A_k^c\right) \leq \sum_{k\geq T}\mu_p(A_k^c). \tag{4.58}$$

It follows from the definition of a measure that

$$\mu_p(A_k^c) \leq \mu_p\left\{(\theta(s)) : |\theta(-k)|_2^2 \geq \tfrac{K_0}{2}|K-1|^\delta\right\} \\ +\mu_p\left\{(\theta(s)) : \sup_{s\in[-k,-k+1]} F_k(s) \geq \tfrac{K_0}{2}|K-1|^\delta\right\}. \tag{4.59}$$

We will now estimate each of the quantities on the right hand side of the above inequality. We proceed with the first one. Chebyeshev's inequality yields

$$\mu_p\left\{(\theta(s)) : |\theta(-k)|_2^2 \geq \frac{K_0}{2}|K-1|^\delta\right\} \leq \frac{16}{K_0^2|k-1|^{2\delta}}(\mathbb{E}|\theta(-k)|_2^2). \tag{4.60}$$

We now use the earlier derived energy estimates to yield

$$\mu_p\left\{(\theta(s)) : |\theta(-k)|_2^2 \geq \frac{K_0}{2}|K-1|^\delta\right\} \leq \frac{C}{|k-1|^{2\delta}}. \tag{4.61}$$

Here C absorbs the uniform bounds of the energy estimates derived earlier. If we choose $\delta > \frac{1}{2}$, then

$$\sum_k \frac{C}{|k-1|^{2\delta}} < \infty. \tag{4.62}$$

Now we have shown that

$$[M_k, M_k] \leq (\sigma^*_{\max})^2 \int_{-k}^{s} |\theta(t)|_2^2 \, \mathrm{d}t. \tag{4.63}$$

We have that

$$F_k(s) \leq C_1 M_k - \frac{1}{(\sigma^*_{\max})^2}[M_k, M_k]. \tag{4.64}$$

Recall the exponential martingale inequality for any positive constants α and β:

$$\mathbb{P}\left\{ \underset{s\in[-k,0]}{sup} M_k(s) - \frac{\alpha}{2}[M_k, M_k] \geq \beta \right\} \leq e^{-\alpha\beta}. \tag{4.65}$$

It therefore follows that

$$\mu_p\left\{(\theta(s)) : \sup_{s\in[-k,-k+1]} F_k(s) \geq \tfrac{K_0}{2}|k-1|^{2\delta}\right\} \\ \leq e^{-\frac{2K_0}{(\sigma_{\max})^2}|k-1|^\delta} + e^{-\frac{2K_0}{(\sigma^*_{\max})^2}|k-1|^\delta} \\ \leq C_1 e^{-C_2|k-1|^\delta}. \tag{4.66}$$

Again, clearly for $\delta > \frac{1}{2}$, we have that

$$\sum_k C_1 e^{-C_2|k-1|^\delta} < \infty. \tag{4.67}$$

Hence we arrive at

$$\sum_{k\geq T} \mu_p(A_k^c) \leq \sum_k \frac{C}{|k-1|^{2\delta}} + \sum_k C_1 e^{-C_2|k-1|^\delta} < \infty. \tag{4.68}$$

This implies that

$$\sum_{k\geq T} \mu_p(A_k^c) < \infty, \tag{4.69}$$

and by the Borel-Cantelli lemma we have that

$$\mu_p(\limsup_{k\to\infty} A_k^c) = 0. \tag{4.70}$$

This tells us that for large values of k the complement of A_k would μ_p-almost never occur. Hence A_k would μ_p-almost certainly occur. Hence there must exist a time T_1 such that

$$|\theta(s)|_2^2 \leq \epsilon_0 + K_0 \min(T_1, |s|)^\delta. \tag{4.71}$$

Therefore the lemma is proved.

5. Control of high modes

We now state a lemma which gives an estimate on the difference of high modes of two different solutions to (2.21). This lemma is crucial for a proof of the third condition that follows subsequently.

Lemma 5.1. *Suppose there exist two solutions to the stochastic infinite Darcy-Prandtl number model:*

$$\theta_1(t) = l(t) + h_1(t) \tag{5.1}$$

and

$$\theta_2(t) = l(t) + h_2(t). \tag{5.2}$$

Then there exists a positive constant C such that when N is chosen so that

$$-\gamma = -(1-\alpha)N^2 + \beta < 0, \tag{5.3}$$

then $\theta_1 = \theta_2$, i.e., $h_1(t) = h_2(t)$. Furthermore, given a solution $\theta(t)$, any h_0, and $t \leq 0$ the limit exists:

$$\lim_{t_0\to-\infty} \Phi_{t_0,t}(l, h_0) = h(t). \tag{5.4}$$

Note the α and β referred to in the Lemma above are introduced in condition 2 in [$E02$]: see the appendix. This Φ is the same as introduced in section 2.3.

Proof. Let $\rho(t) = h_1(t) - h_2(t)$. Then subtracting the requisite equations yields

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -A\rho + P_h[R(\theta_1 - \theta_2)]. \tag{5.5}$$

We multiply the above equation by ρ, integrate by parts and use condition 2 to yield

$$\begin{aligned}
\frac{\mathrm{d}|\rho|_2^2}{\mathrm{d}t} &= -\langle A\rho, \rho\rangle + \langle [R(\theta_1 - \theta_2)], \rho\rangle \\
&\leq -\langle A\rho, \rho\rangle + \frac{1}{4}\langle A\rho, \rho\rangle + (C + K(\theta_1)|\rho|_2^2) \\
&\leq \frac{-3N^2}{4}|\rho|_2^2 + (C + K(\theta_1)|\rho|_2^2.
\end{aligned}$$

Then there exists T_2 depending on t and θ_1 such that for $t_0 \le T_2$, we have that

$$-\left(\frac{-3N^2}{4}\right)(t-t_0) + \int_{t_0}^{t} (C + K(\theta_1))\, \mathrm{d}s \le -\gamma(t-t_0). \tag{5.6}$$

Hence, by Gronwall's inequality, we have

$$\begin{aligned} |\rho|_2^2 &\le \exp\left\{-(\frac{-3N^2}{4})(t-t_0) + \int_{t_0}^{t} (C + K(\theta_1)\, \mathrm{d}s)\right\} \\ &\le e^{-\gamma(t-t_0)} |\rho(t_0)|_2^2 \\ &\le e^{-\gamma(t-t_0)} [\epsilon_0^\theta + K_0(t_0)^\delta]. \end{aligned}$$

It follows then that for any time $t_0 \le \min(T_1, T_2)$, we have, as $t_0 \to -\infty$, exponential decay when $N^2 \ge \frac{4}{3(t-t_0)} \int_{t_0}^{t} (C + K(\theta_1)\, \mathrm{d}s$.

For the second part of the lemma, let the high mode of the given solution $\theta(t)$ be h_1, and the solution to the high-mode equation starting from t_0 and h_0 be h_2. Then we have

$$|\rho|_2^2 \le |h(t_0) - h_0|_2^2 \exp\left\{-\left(\frac{3N^2}{4}\right) + \int_{t_0}^{t} K(\theta_1(s))\, \mathrm{d}s\right\}. \tag{5.7}$$

By the argument made before, $\rho(t)$ decays to 0 as $t_0 \to -\infty$, and the limit equals $h(t)$. This proves the lemma.

5.1. *Proof of third condition*

We are now in a position to verify the third condition from [E02]. We fix $L^0 \in P$ and $\bar{h}(0)$, which is an initial value for the high mode at time 0. Let $L^s = S_s^\omega L^0$ define S_s^ω, and $l(s) = L^t(s)$ for $s \le t$. Then we know that with probability 1, $h(s) = \Phi_s(L^s)$ where $\theta(s) = (l(s), h(s))$, by Lemma 4.3. Fix a constant C_0 such that $|\theta(0)|_2^2 + |\nabla\theta(0)|_2^2 \le C_0$. For any positive C we define

$$D(C) = \left\{ f \in C([0,\infty), L_l^2) : |\theta|_2^2 + |\nabla\theta|_2^2 + \int_0^t (|\nabla\theta|_2^2 + |\Delta\theta|_2^2)\, \mathrm{d}s \le C_1 + C_2\, t + C\, t^\delta \right\}. \tag{5.8}$$

Here $\theta^* = f(s) + \Phi_s(f, \Phi_0(L_0))$

Now we project $\theta(t)$ onto H_l, and by Lemma 5.1 we have that for any $a \in (0,1)$, there exists a C such that

$$\mathbb{P}\left\{\omega : S_s^\omega L^0 \in D(C)\right\} > 1 - a > 0. \tag{5.9}$$

Therefore if we set $\bar{h}(s) = \Phi_s(L^s, \bar{h}(0))$ and $\rho(s) = h(s) - \bar{h}(s)$, then $\theta = l + h = l + \bar{h} + \rho$. We state the following lemma.

Lemma 5.2. *For the set $D(C)$ as defined in* (5.8)*, the following estimate holds*

$$\sup_{\{\omega : S_t^\omega L^0 \in D(C)\}} \int_0^\infty |D(l(t), h(t), \bar{h}(t))|^2\, \mathrm{d}t < \infty, \tag{5.10}$$

where $D(l, h, \bar{h})$ is defined in equation (2.36).

Proof. We have

$$\begin{aligned}
&|D(l(t), h(t), \bar{h}(t)|_2^2 \\
=& \sup_{\{w\in L^2, |w|_2\leq 1\}} |Ra_D \left\langle P_l(P(\mathbf{k}(l+h)\cdot\nabla(l+h)) - P(\mathbf{k}(l+\bar{h})\cdot\nabla(l+\bar{h}))), w\right\rangle|^2 \\
\leq& \sup_{\{w\in L^2, |w|_2\leq 1\}} |Ra_D \ \langle P_l(P(\mathbf{k}(l+h)\cdot\nabla\rho)), w\rangle|^2 \\
&\quad + \sup_{\{w\in L^2, |w|_2\leq 1\}} |\langle Ra_D \ P_l(P(\mathbf{k}\nabla(l+h)\rho)), w\rangle|^2 \\
\leq& \sup_{\{w\in L^2, |w|_2\leq 1\}} |Ra_D \ \langle P_l(P(\mathbf{k}(l+h)\cdot\rho)), \nabla w\rangle|^2 \\
&\quad + \sup_{\{w\in L^2, |w|_2\leq 1\}} |\langle Ra_D \ P_l(P(\mathbf{k}(l+h)\rho)), \nabla w\rangle|^2 \\
\leq& \sup_{\{w\in L^2, |w|_2\leq 1\}} Ra_D^2 \ |P_l(\nabla w)|_\infty^2 |P(\mathbf{k}(l+h))|_2^2 |\rho|_2^2 \\
&\quad + \sup_{\{w\in L^2, |w|_2\leq 1\}} Ra_D^2 \ |P_l(\nabla w)|_\infty^2 |P(\mathbf{k}(l+h))|_2^2 |\rho|_2^2 \\
\leq& \sup_{\{w\in L^2, |w|_2\leq 1\}} Ra_D^2(|w|_{H^3}^2)|\theta|_2^2|\rho|_2^2 \\
\leq& \, C(N) \ Ra_D^2 |\theta|_2^2 |\rho|_2^2.
\end{aligned}$$

This follows via integration by parts and the compact embedding

$$H^2(X) \hookrightarrow L^\infty(X) \hookrightarrow L^2(X). \tag{5.11}$$

Note if $L^t \in D(C)$ we can use the estimates on $|\theta|_2^2$ in Lemma 4.2, and on $|\rho|_2^2$ in Lemma 5.1 to yield

$$|D(l(t), h(t), \bar{h}(t)|_2^2 \leq |\rho(0)|_2^2 C(N) e^{(-N^2+\frac{1}{2}+C)t}(C_1 + C_2 t). \tag{5.12}$$

If N is chosen such that $N^2 > \frac{1}{2} + C$ as $t \to \infty$, the right hand side decays exponentially. This yields

$$\begin{aligned}
\sup_{\{\omega: S_t^\omega L^0 \in D(C)\}} &\int_0^\infty |D(l(t), h(t), \bar{h}(t))|_2^2 \, \mathrm{d}t \\
&\leq \sup_{\{\omega: S_t^\omega L^0 \in D(C)\}} \int_0^\infty t^\delta C(N) e^{(-N^2+\frac{1}{2}+C)t}(C_1 + C_2 t) \, \mathrm{d}t \\
&< K \\
&< \infty.
\end{aligned}$$

This proves the lemma. □

Thus condition 3 (see appendix) from [$E02$] is verified.

5.2. *Estimation of high modes*

The following lemma will be useful in proving the last condition from [$E02$]. This gives an estimate on the high modes of a solution to the stochastic infinite Darcy-Prandtl number model.

Lemma 5.3. *If $h(t)$ is the high mode component of the solution to the stochastic Darcy-Boussinesq equation with low-mode forcing $l \in C([0,t], L^2_l)$, then $|h(t)|_2^2$ is bounded by a constant that depends only on $|h(0)|_2^2$ and $\int_0^t |l(s)|_2^2 \, \mathrm{d}s$.*

Proof. We start with the equations for the high modes of the system, multiply them by the high mode component and integrate the result by parts to yield

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|h|_2^2 &\le -2|\nabla h|_2^2 - 2Ra_D P_h(P(\mathbf{k}(l+h)) \cdot \nabla(l+h)h + 2Ra_D\ \gamma'(z)P_h[P(l+h)]h \\
&\le -2|\nabla h|_2^2 - 2Ra_D\ \langle P(\mathbf{k}(l+h) \cdot \nabla l, h\rangle - \langle P(\mathbf{k}(l+h)\cdot \nabla h, h\rangle + C\ |h|_2^2 \\
&\le -2|\nabla h|_2^2 - 2Ra_D \int_H (ll_z h + hl_z h)\,\mathrm{d}\mathbf{x} + 2Ra_D|h|_2^2 \\
&\le Ra_D|l_z|_\infty|l|_2^2 + Ra_D|l_z|_\infty|h|_2^2 + 2Ra_D|l_z|_\infty|h|_2^2 + C\ |h|_2^2 \\
&\le Ra_D|\Delta l_z|_2|l|_2^2 + Ra_D|\Delta l_z|_2|h|_2^2 + 2Ra_D|\Delta l_z|_2|h|_2^2 + C\ |h|_2^2 \\
&\le Ra_D C(N)|l|_2^2 + (3C(N)Ra_D + C)|h|_2^2 \\
&\le Ra_D C(N)|l|_2^2 + C(N)Ra_D|h|_2^2.
\end{aligned}$$

These follow via integration by parts, the embedding $H^2(X) \hookrightarrow L^\infty(X)$ and the uniform estimate on $|\theta|_2$. Thus, via the Gronwall inequality, we obtain

$$|h|_2^2 \le |h(0)|_2^2 + \int_0^t C(N)Ra_D|l|_2^2 \exp\left\{C\int_0^s |l(\tau)|_2^2\,\mathrm{d}\tau + C(N)Ra_D s\right\}\mathrm{d}s. \quad (5.13)$$

This completes the proof of the lemma. □

5.3. *Proof of last condition*

Let us assume that $|l(0)|_2 \le M$. We define, for $f \in C([0,\infty), L^2_l)$ and $0 \le s \le T$, $\theta(s) = f(s) + \Phi_s(f, \Phi_0(L^0))$. Set

$$D_T(C) = \left\{ f \in C([0,\infty), L^2_l) : \int_0^t |\theta|_2^2\,\mathrm{d}s < CT, 0 \le t \le T \right\}. \quad (5.14)$$

We now state the following lemma.

Lemma 5.4. *For the set $D_T(C)$ defined in* (5.14)*, the following estimate holds:*

$$\sup_{\{L^t \in D_T\}} \int_0^t |G(l(s), \Phi_s(L^s, \Phi_0(L^0))|_2^2\,\mathrm{d}s < \infty. \quad (5.15)$$

Proof. We know $h(s) = \Phi_s(L^s, \Phi_0(L^0))$ with probability 1, from Lemma 5.1. Thus we have

$$\begin{aligned}
&|G(l(s), \Phi_s(L^s, \phi_0(L^0)))|_2 \\
&\quad = \sup_{\{w \in L^2, |w|_2 \leq 1\}} \langle P_l(-P(\mathbf{k}(l+h)) \cdot \nabla(l+h)) + Ra_D \; P_l(l+h)_3, w\rangle \\
&\quad \leq \sup_{\{w \in L^2, |w|_2 \leq 1\}} \frac{1}{2} \left\langle P_l(P(\nabla(l+h)))^2, w \right\rangle + \sup_{\{w \in L^2, |w|_2 \leq 1\}} \langle Ra_D \; P_l(l+h), w\rangle \\
&\quad \leq \sup_{\{w \in L^2, |w|_2 \leq 1\}} |P_l \nabla w|_\infty \langle |l+h|, |l+h|\rangle + Ra_D \; |P_l \nabla w|_2 |l|_2 \\
&\quad \leq \sup_{\{w \in L^2, |w|_2 \leq 1\}} |\nabla w|_\infty (|l+h|_2^2 + Ra_D \; |l|_2^2) \\
&\quad \leq C(N)(|l|_2^2 + |h|_2^2).
\end{aligned}$$

Here $C(N)$ absorbs various constants. Therefore we have that for $l \in D_T(C^*)$

$$\begin{aligned}
|h|_2^2 &\leq |h(0)|_2^2 + \int_0^t C(N) Ra_D |l|_2^2 \exp\left\{C \int_0^s |l(\tau)|_2^2 \, \mathrm{d}\tau + (C(N) Ra_D) s\right\} \mathrm{d}s \\
&\leq |h(0)|_2^2 + \int_0^T C(N) Ra_D |l|_2^2 \exp\left\{C \int_0^s |l(\tau)|_2^2 \, \mathrm{d}\tau + (C(N) Ra_D) s\right\} \mathrm{d}s.
\end{aligned}$$

This is less than some C by Lemma 5.3, as we are on a finite time interval $[0, T]$. Here C depends only on $|h_0|_2$ and the C and T used to define $D_T(C)$. Therefore

$$\begin{aligned}
&\sup_{L^t \in D_T} \int_0^T |G(l(s), \Phi_s(L^s, \Phi_0(L^0))|_2^2 \, \mathrm{d}s \\
&\quad \leq \int_0^T C(N)(|l|_2^2 + |h|_2^2) \, \mathrm{d}s \\
&\quad \leq C(N)CT + \int_0^T C(N) Ra_D |l|_2^2 \exp\left\{C \int_0^t |l(s)|_2^2 \, \mathrm{d}s + (C(N) Ra_D) t\right\} \mathrm{d}t \\
&\quad \leq C_{12} \; T + C_{13} e^{C_{14} T} \\
&\quad \leq K \\
&\quad < \infty.
\end{aligned}$$

Thus the Lemma is proved. □

Condition 4 (see appendix) from [E02] now follows. Since we have verified that the stochastic infinite Darcy-Prandtl number model satisfies all the conditions as postulated in [E02], we can apply the results that appear therein, to yield the following result:

Theorem 5.1. *The stochastic infinite Darcy-Prandtl number model, as defined via* (2.21)*, possesses a unique invariant measure for*

$$N^2 > \frac{4C}{3}, \tag{5.16}$$

where C is the uniform constant appearing in Lemma 4.2.

6. Conclusion

In conclusion we have shown the uniqueness of invariant measure for the stochastic infinite Darcy-Prandtl number model, under stochastic forcing of low modes. Various questions remain open at this point, for example, the uniqueness of an invariant measure for the stochastic Darcy-Boussinesq system. In this case we would have to add noise to both the velocity and temperature equations. This is conceivably a harder question to tackle as the Darcy-Boussinesq system is only weakly dissipative.

We have shown in $[P10]$ that the stationary statistical properties for the deterministic Darcy-Boussinesq system are upper semi-continuous after lifting in the singular limit, i.e., for $\mu_\epsilon \in IM_\epsilon$, $0 \leq \epsilon \leq \epsilon_0$, there exists a weakly convergent subsequence, denoted μ_ϵ, and $\mu_0 \in IM_0$ such that

$$\mu_\epsilon \rightharpoonup L(\mu_0). \tag{6.1}$$

Here μ_ϵ is an invariant measure for the Darcy-Boussinesq system and μ_0 is an invariant measure for the infinite Darcy-Prandtl number model. It would be of further interest then to derive uniqueness conditions on the μ_ϵ, and μ_0 as well.

Another difficult question to consider would be the zero-noise limit of both the stochastic infinite Darcy-Prandtl number model and stochastic Darcy-Boussinesq system. In particular since we have shown uniqueness for the stochastic infinite Darcy-Prandtl number model, we could consider a "δ" model of the stochastic infinite Darcy-Prandtl number model, where δ is a simple multiplicative parameter which enters the system as

$$\mathrm{d}\theta = (\Delta\theta - P(\mathbf{k}\theta)\cdot\nabla\theta + Ra_D P(\theta))\,\mathrm{d}t + \delta\,\mathrm{d}\mathbb{W}, \tag{6.2}$$

$$\theta|_{z=0} = 0, \quad \theta|_{z=1} = 0. \tag{6.3}$$

Hence we would have a unique invariant measure for each fixed δ. What can be said about the limit of these as δ goes to zero, the "zero noise limit"? Some work has been done in this regard for axiom A systems $[Y02]$. However these are difficult to verify pragmatically for most physical systems we deal with.

7. Appendix

Consider the following stochastic PDE

$$\mathrm{d}u = (\Delta u + R(u))\,\mathrm{d}t + \mathrm{d}\mathbb{W}, t \geq 0, u_0 = u(0). \tag{7.1}$$

The following conditions are imposed on (7.1)

Condition 1: There exists constants $\eta \geq 0$ and $K_0 \geq 0$ such that

$$-\langle Ax, x\rangle_H + \langle Rx, x\rangle_H \leq -\eta|x|_H^2 + K_0. \tag{7.2}$$

Condition 2: Let $\theta_1, \theta_2 \in H$ and let $\rho = \theta_1 - \theta_2$. There exists a constant $\alpha \in [0,1)$ and a non-negative function $K(\theta)$ on H such that

$$\langle R(\theta_1) - R(\theta_2), \rho\rangle_H \leq \alpha\,\langle A\rho, \rho\rangle_H + K(\theta_1)|\rho|_H^2. \tag{7.3}$$

Furthermore

$$\int_H K(\theta)\,\mathrm{d}\mu(\theta) \le \beta, \tag{7.4}$$

for some constant β independent of the invariant measure μ.

Condition 3: For all $L^0 \in P$, $\bar{h}_0$, and for all $a \in (0,1)$ and $T \ge 0$, there exists $K \ge 0$ such that

$$\mathbb{P}\left\{\int_0^\infty |D(l(t), h(t), \bar{h}(t))|_H^2\,\mathrm{d}s < K\right\} \ge 1 - a > 0. \tag{7.5}$$

Condition 4: [a] For all $L^0 \in P$, $a \in (0,1)$ and $T \ge 0$, there exists $K \ge 0$ such that

$$\mathbb{P}\left\{\int_0^T |G(l(s), h(s))|_H^2\,\mathrm{d}s < K\right\} \ge 1 - a > 0. \tag{7.6}$$

Theorem 7.1. *(E and Liu, Journal of Statistical physics, 2002) Suppose that the stochastic partial differential equation* (7.1) *satisfies conditions 1–4 and N is chosen large enough, then equation* (7.1) *has a unique invariant measure.*

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References

CD96. C. Doering and P. Constantin, *heat transfer in Convective turbulence*, Nonlinearity., **9** (1996), 1049-1060.

DZ96. G. Da Prato and J. Zabczyk, “Ergodicity for Infinite Dimensional Systems”, London Mathematical Society Lecture Note Series, **229**, Cambridge University Press, Cambridge, 1996.

DC98. C. Doering and P. Constantin, *Bounds for heat transpport in a porous layer*, J. Fluid Mech., **376** (1998), 263-296.

E97. L.C.Evans, “Partial Differential Equations”, Graduate Studies in Mathematics, **19**, Providence, 1998.

L48. E.R. Lapwood, *Convection of a fluid in a porous medium*, Procl. Camb. Phil. Soc, **44** (1948), 508-521.

E02. D.Liu and W.E, *Gibbsian dynamics and invariant measures for stochastic PDE*, J. Stat. Phys, **108** (2002),1125-1156.

M98. J.Mattingly, “The stochastically forced Navier-Stokes equations: Energy estimates and phase space contractions”, Ph.d thesis,1998, Princeton University

MES01. J.Mattingly, E.Weinan, Y.Sinai, “Gibbsian Dynamics and ergodicity for the stochastically forced Navier-Stokes equations”, Communications in Mathematical Physics, **224**, (2001), 83106

[a] For the notation in conditions 3 and 4 please refer to section 2.3

HR45. C.W.Horton and F.T. Rogers, *Convection Currents in a porous medium*, J. Appl. Phys (1945), 367-370.

LT99. H.V. Ly and E.S. Titi, *Global gevry regularity for the 3D benard convection in porous medium with zero darcy-prandtl number*, J Nonlinear Sci (1999), 333-362.

N99. D. Nield and A. Bejan, "Convection in Porous Media", 2^{nd} ed, Springer-Verlag, New York, 1999.

P10. R.D. Parshad, *Asymptotic Behaviour of the Darcy-Boussinesq System at Large Darcy-Prandtl Number*, Discrete & Contin Dyn Sys-A, Special issue 18, Vol26, no.4, 1441-1469, (2010).

PT09. R.D. Parshad, Asymptotic Behaviour of Convection in Porous Media, PhD Thesis, Florida State University, 2009.

R00. R.J.Charbeneau, "Groundwater Hydraulics and Pollutant Transport", Prentice Hall, Upper Saddle River, 2000.

S93. M.Saad, *Ensembles inertiels pour un modele de convection naturelle dissipatif, en milieu poreux*, C.R. Acad. Sci Paris, **Serie I 316** (1993), 1277-1280.

S01. J. Saling, "Radioactive waste management", Taylor and Francis, New York, 2001.

S09. J. Speight, "Enhanced Oil Recovery Handbook", Gulf Publishing Company, 2009.

T97. R. Temam, "Infinite-dimensional dynamical systems in mechanics and physics". 2^{nd} ed. Applied Mathematical Sciences, 68. Springer, New York, 1997.

WS09. X.Wang, "Lecture notes on elementary statistical theories with applications to fluid systems", 2007 Shanghai Mathematics Summer School in Fudan University , To be published by Higher Education Press, 2009.

Wu04. M. Wu and J. Lee, *Ergodicity for the dissipative boussinesq equations with random forcing*, J Stat Phys, **117** (2004), 929-957.

Y02. L.S.Young, *What are SRB measures and which dynamical systems have them?*, J Stat Phys, **108** (2002), 733-754.