Remarks about homework problem No. 4, Chap. IV

Contour of integration: No need to consider squares or anything. You can use the standard segment $[-R, R]$ on the real line followed by a closing circle. Which circle (in the lower or upper) half plane will depend on the sign of ξ, since

$$e^{-2\pi i\xi z} = e^{-2\pi i\xi x} e^{2\pi\xi y}.$$

So the three cases $\xi > 0$, $\xi < 0$, and $\xi = 0$ will need to be considered separately.

Calculation: Use residues, plus prove that the part of the integral on the upper (or lower) circle will go to zero as $R \to \infty$. This requires an analysis of the denominator.

Residues: Contrary to what I claimed in class, no need to assume the denominator $Q(x)$ is real, although it causes no harm to do so. Clearly, the parameter R will have to be big enough so that all the roots are enclosed.

For the calculation, you can either use that

$$Q(z) = c \prod_{k=1}^{n} (z - \lambda_k),$$

and c is a non-zero complex number, or for any root λ define

$$Q_\lambda(z) = \frac{Q(z)}{z - \lambda},$$

so, as explained in class, the residue at $z = \lambda$ will be:

$$\frac{e^{-2\pi i\xi\lambda}}{Q_\lambda(\lambda)}.$$

Note that if you want to assume that $Q(z)$ is a real polynomial, since there are no real roots, then the degree n is even, say $n = 2m$, and all the roots will come in pairs as $\lambda_k, \bar{\lambda}_k = \pm i a_k$, with all $a_k > 0$, for $k = 1, \dots, m$.