Quiz 3/**test prep 2** Name: *(10 pts)* ~ KEY ~

(please print neatly!)

Directions: Answer each of the following **two (2)** questions, making sure to read the instructions for each question as you proceed.

You may use the backs of the pages for scratch work **or** get scrap paper from me!

The last page is a bonus problem! Make sure you don't overlook it!

1. *(20 pts)* Solve the IVP

$$y'' + 2y' + y = 3e^{-x}, \quad y(0) = 1, \quad y'(0) = 1.$$

Hint: $(x^2e^{kx})' = kx^2e^{kx} + 2xe^{kx}$ and $(x^2e^{kx})'' = k^2x^2e^{kx} + 4kxe^{kx} + 2e^{kx}$ for all constants k.

SOLUTION: Hom: $y''+2y'+y=0 \Leftrightarrow r^2+2r+1 \overset{=0}{\Leftrightarrow} (r+1)(r+1)=0 \Leftrightarrow r=-1,\ r=-1$

$\hookrightarrow c_1e^{-x} + c_2xe^{-x}$

Non-Hom: $Y(x) = Ax^2e^{-x} \rightsquigarrow Y' = -Ax^2e^{-x} + 2Axe^{-x}$

$Y'' = Ax^2e^{-x} - 2Axe^{-x} - 2Axe^{-x} + 2Ae^{-x}$

$$\Rightarrow Y''+2Y'+Y = \left.\begin{array}{l} Ax^2e^{-x} - 4Axe^{-x} + 2Ae^{-x} \\ + -2Ax^2e^{-x} + 4Axe^{-x} \\ + Ax^2e^{-x} \end{array}\right] = 3e^{-x}$$

$\Rightarrow 2A = 3$

$\Rightarrow A = 3/2$ $\rightsquigarrow$ Gen Soln: $\boxed{y = c_1e^{-x} + c_2xe^{-x} + \frac{3}{2}x^2e^{-x}}$

IVP: • $y(0)=1 \Rightarrow c_1 = 1.$

• $y' = -c_1e^{-x} - c_2xe^{-x} + c_2e^{-x} - \frac{3}{2}x^2e^{-x} + 3xe^{-x} \Rightarrow 1 = y'(0) = -c_1 + c_2$

$\Rightarrow 1 + c_1 = c_2 \Rightarrow \boxed{c_2 = 2}$

$$\boxed{\text{Ans: } y = e^{-x} + 2xe^{-x} + \frac{3}{2}x^2e^{-x}}$$

2. (a) (*6 pts*) **True or False:** $y_1(t) = e^{-2t}$ and $y_2 = e^{3t}$ are both solutions to the second-order homogeneous ODE

$$y'' - y' - 6y = 0.$$

Justify your claim! $\boxed{\text{TRUE}}$

SOLUTION:

$y_1'' - y_1' - 6y_1 = 4e^{-2t} + 2e^{-2t} - 6e^{-2t}$

$= 0$

is a soln

$y_2'' - y_2' - 6y_2 = 9e^{3t} - 3e^{3t} - 6e^{3t}$

$= 0$

is a soln

(b) (*4 pts*) **True or False:** $y_1(t) = e^{-2t}$ and $y_2 = e^{3t}$ form a fundamental system of solutions for the second-order homogeneous ODE

$$y'' - y' - 6y = 0.$$

Justify your claim! $\boxed{\text{TRUE}}$

SOLUTION: $W(e^{-2t}, e^{3t}) = \det\begin{pmatrix} e^{-2t} & e^{3t} \\ -2e^{-2t} & 3e^{3t} \end{pmatrix}$

$= 3e^{t} + 2e^{t}$

$= 5e^{t}.$

(a) + $W \neq 0 \Rightarrow y_1$ & y_2 are F.S.S!

(c) (*10 pts*) *Without using undetermined coefficients*, use parts (a) and (b) to find a particular solution of the non-homogeneous ODE

$$y'' - y' - 6y = e^{-3t}.$$

Simplify fully.

SOLUTION: $g(t) = e^{-3t}$ $\quad y_1 = e^{-2t}$ $\quad y_2 = e^{3t}$ $\quad W(y_1, y_2) = 5e^{t}$

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1,y_2)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)}\,dt$$

$$= -e^{-2t}\int \frac{e^{3t}(e^{-3t})}{5e^{t}}\,dt + e^{3t}\int \frac{e^{-2t}(e^{-3t})}{5e^{t}}\,dt$$

$$= \frac{-e^{-2t}}{5}\int e^{-t}\,dt + \frac{e^{3t}}{5}\int e^{-6t}\,dt$$

$$= \frac{e^{-3t}}{5} - \frac{1}{30}e^{-3t}$$

$$= \frac{5}{30}e^{-3t} = \frac{1}{6}e^{-3t}$$

Bonus: (*5 pts*) Use the method of undetermined coefficients to check the answer you got for problem 2(c).

In other words: Use the method of undetermined coefficients to find a particular solution of the non-homogeneous ODE

$$y'' - y' - 6y = e^{-3t}.$$

Hint: When using variation of parameters, some of the terms produced in the formula for $Y(t)$ may have the form (const)y_1 or (const)y_2; to get that answer to match the one obtained using undetermined coefficients, you want to <u>ignore</u> any such terms! (make sure you know why!)

SOLUTION:

$Y(t) = Ae^{-3t}$ $\quad Y'(t) = -3Ae^{-3t}$ $\quad Y''(t) = 9Ae^{-3t}$

$$e^{-3t} = Y'' - Y' - 6Y$$
$$= 9Ae^{-3t} + 3Ae^{-3t} - 6Ae^{3t}$$
$$= 6Ae^{3t}$$

$\Rightarrow \quad 1 = 6A$

$\Rightarrow \quad A = \frac{1}{6}$

So $Y = \frac{1}{6}e^{-3t}$, as before.