§6.2. – Solutions to IVP (using Laplace)

Recall: If $f(t)$ is piecewise continuous and grows no faster than $(\text{const})\cdot e^{t}$, then the Laplace transform of f is the function

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt. \quad \left(= \text{"}\mathcal{L}\{f(t)\}\text{"}\right)$$

Facts:
- Laplace transforms <u>aren't</u> unique (e.g. if f has discontinuity)
- Laplace transforms are "linear operators":

$$\mathcal{L}\{a\cdot f(t)+b\cdot g(t)\} = \int_0^\infty e^{-st}\left(a\cdot f(t)+b\cdot g(t)\right)dt$$

$$= \int_0^\infty e^{-st}\cdot a\cdot f(t)\,dt + \int_0^\infty e^{-st}\cdot b\cdot g(t)\,dt$$

$$= a\cdot\int_0^\infty e^{-st}f(t)\,dt + b\cdot\int_0^\infty e^{-st}g(t)\,dt$$

$$= a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}.$$

Goal: To use Laplace Transforms to solve IVPs!

↓

Thm: Suppose Laplace exists for $f(t)$ & that $f'(t)$ is piecewise continuous on $[0,\infty)$. Then $\mathcal{L}\{f'(t)\}$ exists, and

$$\mathcal{L}\{f'(t)\} = s\cdot\mathcal{L}\{f(t)\} - f(0).$$

<u>why?</u> $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt \overset{\text{IBP}}{=} \boxed{e^{-st}f(t)\Big]_{t=0}^{t=\infty}} - \int_0^\infty (-s)e^{-st}f(t)\,dt$

$= \boxed{0 - f(0)} + s\int_0^\infty e^{-st}f(t)\,dt = 0 - f(0) + s\,\mathcal{L}\{f(t)\}$

$= s\,\mathcal{L}\{f(t)\} - f(0).$

Corollary

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

How is this helpful?

Ex: $y'' - y' - 2y = 0$ $\longrightarrow$ $\boxed{\text{Solve old way:}}$ $r^2 - r - 2 = 0$

$y(0)=1 \quad y'(0)=0$

$(r-2)(r+1)=0$

$r=2,\ r=-1$

$\Rightarrow$ gen soln: $y = c_1 e^{2x} + c_2 e^{-x}$

$\Rightarrow$ I.V.: • $1 = c_1 + c_2$

• $y' = 2c_1 e^{2x} - c_2 e^{-x}$

$\Rightarrow 0 = 2c_1 - c_2$

$\Leftarrow$

So: • $3c_1 = 1 \Rightarrow c_1 = \frac{1}{3}$

• $c_2 = \frac{2}{3}$

Part. Soln: $y = \frac{1}{3}e^{2x} + \frac{2}{3}e^{-x}$

$\boxed{\text{Take Laplace Instead}}$

$$y'' - y' - 2y = 0 \Rightarrow \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\Rightarrow \left(s^2\mathcal{L}\{y(t)\} - s\overset{1}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}}\right)$$

$$- \left(s\mathcal{L}\{y(t)\} - \overset{1}{\cancel{y(0)}}\right) - 2\mathcal{L}\{y\} = 0$$

$$\Rightarrow \mathcal{L}\{y(t)\}\left(s^2 - s - 2\right) - s + 1 = 0$$

$$\Rightarrow \mathcal{L}\{y(t)\} = \frac{s-1}{s^2-s-2}$$

$\Rightarrow$ The solution to the ODE is the func/expr. y whose Laplace satisfies this!

Ex (Cont'd)

Use partial fractions:

$$\frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$\Rightarrow s-1 = A(s+1) + B(s-2)$$

$$\Rightarrow \begin{cases} -2 = B(-3) \Rightarrow B = 2/3 \\ 1 = A(3) \Rightarrow A = 1/3 \end{cases}$$

So,

$$\mathcal{L}\{y(t)\} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

look for these in mid. column of Laplace Table

$$\Downarrow$$

$$y(t) = \text{"}\mathcal{L}^{-1}\left(\frac{1/3}{s-2}\right)\text{"} + \text{"}\mathcal{L}^{-1}\left(\frac{2/3}{s+1}\right)\text{"}$$

(inverse Laplace... we don't know this notation yet.)

$$= \tfrac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

You get the same answer doing alg. instead of calc!

Summarize

- Start w/ IVP
- Take Laplace
- Plug in

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

- Plug in $f(0)$ & $f'(0)$
- Isolate $\mathcal{L}\{f(t)\}$ on LHS
- Find "inverse Laplace" of RHS, i.e. the functions whose Laplaces equal RHS.

using Laplace table, which I'll give on the exam!

- This works for nonhomogeneous IVPs too!

Ex: $y'' + y = \sin(2t)$, $y(0) = 2$, $y'(0) = 1$.

$$\Rightarrow \left(s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)\right) + \mathcal{L}\{y(t)\} = \mathcal{L}\{\sin(2t)\}$$

$$\Rightarrow s^2 Y(s) - 2s - 1 + Y(s) = \frac{2}{s^2+4}$$ from table

$$\Rightarrow Y(s)(s^2+1) = \frac{2}{s^2+4} + 2s + 1$$

$2s^3 + s^2 + 8s + 4 + 2$

$$\Rightarrow Y(s)(s^2+1) = \frac{2 + (2s+1)(s^2+4)}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+1)} \xrightarrow[\text{fracs}]{\text{partial}} \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

Ex (Cont'd)

So: $2s^3+s^2+8s+6 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$

$\Rightarrow 2s^3+s^2+8s+6 = As^3 + As + Bs^2 + B + Cs^3 + 4Cs + Ds^2 + 4D$

$\Rightarrow A+C=2 \qquad B+D=1$

$A+4C=8 \qquad B+4D=6$

$3C=6 \Rightarrow C=2 \qquad 3D=5 \Rightarrow D=\frac{5}{3}$

$\Rightarrow A=0 \qquad \Rightarrow B=\frac{-2}{3}.$

$$\Rightarrow Y(s) = \frac{-2}{3}\left(\frac{1}{s^2+4}\right) + \frac{2s+5/3}{s^2+1}$$

$$= \frac{-1}{3}\underbrace{\left(\frac{2}{s^2+4}\right)}_{} + 2\underbrace{\left(\frac{s}{s^2+1}\right)}_{} + \frac{5}{3}\underbrace{\left(\frac{1}{s^2+1}\right)}_{}$$

"Looks like" $\frac{a}{s^2+a^2}$, $a=2$ $\qquad \frac{s}{s^2+a^2}$, $a=1$ $\qquad \frac{a}{s^2+a^2}$, $a=1$

using Laplace Table $\rightarrow$

$$\Rightarrow y(t) = \frac{-1}{3}\sin(2t) + 2\cos(t) + \frac{5}{3}\sin(t)$$