Note: An alternative (hopefully less-confusing) version of the §3.6 formula is as an INDEFINITE integral:

$$Y(t) = -y_1(t)\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)}\,dt + y_2(t)\int \frac{y_1(t)\,g(t)}{W(y_1,y_2)}\,dt. \qquad (\star)$$

This (a) avoids the confusing "s" variables, (b) doesn't make mention of I, and (c) doesn't require a t_0 and/or plugging in any bounds. Per the theorem, ($\star$) is valid on any open interval on which p, q, and g are all continuous, where

$$y'' + p(t)y' + q(t)y = g(t)$$

and where y_1 & y_2 are a F.S.S. of the corresponding homogeneous ODE.

§6.1 - The Laplace transform

Def: Given a function $f(t)$, the Laplace Transform of f is the function $\mathcal{L}\{f(t)\} = F(s)$ given by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad \text{assuming this converges.}$$

Note: The thing you plug in is a function of t ("time," a real variable) and the thing you get out is a function of s ("frequency," a complex variable).

Ex: Let $f(t) = 1$ ($t \geq 0$). Then

$$F(s) = \int_0^\infty e^{-st}(1)\,dt = \int_0^\infty e^{-st} = \frac{-1}{s}\left[e^{-st} \Big]_{t=0}^{"t=\infty"} \right]$$

$$= \frac{-1}{s}\left[0 - 1\right] = \frac{1}{s}. \quad (s > 0)$$

Note: Technically, $\int_0^\infty e^{-st}\,dt = \lim_{k\to\infty}\int_0^k e^{-st}\,dt = \lim_{k\to\infty}\left[\frac{-1}{s}\left(e^{-st}\Big]_{t=0}^{t=k}\right)\right] \cdots$

Ex: $f(t) = e^{at}$, $t \geq 0$.

$$F(s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{(a-s)t}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \frac{1}{a-s}\left(e^{-(s-a)t} \Big]_{t=0}^{``t=\infty"} \right)$$

$$= \frac{1}{a-s}(0-1) = \frac{-1}{a-s} = \frac{1}{s-a}.$$

$(s > a)$

Ex: $f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ K & t = 1 \\ 0 & t > 1 \end{cases}$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}\, dt + \int_1^1 Ke^{-st}\, dt + \int_1^\infty 0\, dt$$

$$= \int_0^1 e^{-st}\, dt + 0 + 0$$

$$= \frac{-1}{s} e^{-st} \Big]_{t=0}^{t=1} = \frac{-1}{s}(e^{-s} - 1).$$

Note: The value K didn't matter! Plugging in any K gives the same result!

↳ True/False: There is only one function f(t) whose laplace Transform is F(s).

FALSE! If f has a discontinuity, There are ∞-many functions w/ the same laplace as f!

we'll see later that under some conditions, Laplaces are unique.

Ex: $f(t) = \sin(bt)$, $t \geq 0$ (b = const).

↳ **Method 1**: $F(s) = \int_0^\infty e^{-st}\sin(bt)\,dt$ &

$$\int e^{-st}\sin(bt)\,dt = -\frac{1}{b}e^{-st}\cos(bt) - \frac{s}{b}\int e^{-st}\cos(bt)\,dt$$

$u = e^{-st}$, $v = -\frac{\cos(bt)}{b}$; $u' = -s e^{-st}$, $v' = \sin(bt)$

$u = e^{-st}$, $v = \frac{1}{b}\sin(bt)$; $u' = -s e^{-st}$, $v' = \cos(bt)$

I found/fixed the sign error here!

$$= \frac{-1}{b}e^{-st}\cos(bt) - \frac{s}{b}\left(\frac{1}{b}e^{-st}\sin(bt) + \frac{s}{b}\int e^{-st}\sin(bt)\,dt\right)$$

$$\Rightarrow \left(\int e^{-st}\sin(bt)\right)\underbrace{\left(1+\frac{s^2}{b^2}\right)}_{\frac{b^2+s^2}{b^2}} = \frac{-1}{b}e^{-st}\left(\frac{s}{b}\sin(bt) + \cos(bt)\right)$$

$$\Rightarrow \int e^{-st}\sin(bt)\,dt = \frac{-b}{b^2+s^2}e^{-st}\left(\frac{s}{b}\sin(bt)+\cos(bt)\right)$$

$$\Rightarrow \int_0^\infty e^{-st}\sin(bt)\,dt = \frac{-b}{b^2+s^2}e^{-st}\left(\frac{s}{b}\sin(bt)+\cos(bt)\right)\Bigg]_{t=0}^{t=\infty}$$

$$= 0 - \frac{-b}{b^2+s^2}(1)(+1)$$

!!

$$= \frac{b}{b^2+s^2} \qquad s > 0.$$

This is correct now!

$$\frac{1}{2i}\left(\frac{s+ib-(s-ib)}{(s-ib)(s+ib)}\right) = \frac{1}{2i}\left(\frac{2ib}{s^2-sib+sib-i^2b^2}\right) = \frac{b}{s^2+b^2}$$

Method 2: Fact: $\sin(bt) = \frac{e^{ibt}-e^{-ibt}}{2i}$ $\quad (i^2=-1)$

$$\Rightarrow F(s) = \int_0^\infty e^{-st}\left(\frac{e^{ibt}-e^{-ibt}}{2i}\right)dt = \frac{1}{2i}\int_0^\infty e^{t(ib-s)} - e^{-t(s+ib)}\,dt$$

$$= \frac{1}{2i}\int_0^\infty e^{-t(s-ib)} - e^{-t(s+ib)}\,dt =$$

$$= \frac{1}{2i}\left(\frac{1}{-(s-ib)}e^{-t(s-ib)} - \frac{1}{-(s+ib)}e^{-t(s+ib)}\Bigg]_{t=0}^{t=\infty}\right) = \frac{1}{2i}\left(0+\frac{1}{s-ib}-\frac{1}{s+ib}\right)$$

(Existence of laplace Thm)

The laplace transform $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ exists for $s > a$

if :

① f is piecewise continuous in $(0,\infty)$; and

② f doesn't "grow faster" than $(\text{const}) \cdot e^{at}$, $a = \text{const}$.

- piecewise continuous means "continuous except for a finite # of jump discontinuities" [no V.A., ...]
- ② means that trig functions, logs, polynomials, $\#^t$, ... all have Laplace Transforms, but things like $f(t) = t^t$, $f(t) = e^{e^t}$ do not.
- As we've seen, Laplace transforms aren't unique in general.

Ex: $f(t) = t^2 \Rightarrow F(s) = \int_0^\infty e^{-st} t^2\,dt = \frac{2}{s^3}$ (prove it)!

$$g(t) = \begin{cases} t^2 & t \neq 7 \\ \text{anything} & t = 7 \end{cases} \Rightarrow G(s) = \int_0^\infty e^{-st} g(t)\,dt = \frac{2}{s^3}$$

may be any #, ∞, DNE, ...

any # can go here!