§5.3 - Part II

- The method we used before involved choosing a power series
$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (x-0)^n$$

Centered at $x_0 = 0$.

- This only ~~[scribbled out]~~ has the chance of working if 0 is an ordinary pt of the ODE.

Def: x_0 is an ordinary point of $P(x)y'' + Q(x)y' + R(x)y = 0$ if the functions
$$p = \frac{Q}{P} \quad \text{and} \quad q = \frac{R}{P}$$
are "analytic" at x_0, i.e. if both have taylor series which converge to them in some interval about x_0.

$f' = -(1+x^2)^{-2} 2x$

$f'' = -2(1+x^2)^{-2} + 2(1+x^2)^{-3} 2x$

Ex: Is 0 an ordinary point of $(1+x^2)^{-1}$?

Fact: Taylor series of $f(x)$ about $x_0 = 0$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$, so:

$$(1+x^2)^{-1} = 1 - 0 - x^2 + 0 + x^4 - 0 - x^6 + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

using Ratio test: Radius of convergence is 1. about $x_0 = 0$.

Ex: If $\frac{Q}{P}$ has a Taylor series which converges on $(-1,4]$ and $\frac{R}{P}$ has a Taylor series which converges on $[-8,3]$, which points are ordinary pts for

$$P(x)y'' + Q(x)y' + R(x)y = 0?$$

Ans: Intersect $(-1,4]$ w/ $[-8,3]$: $(-1,3]$.

Ex 2: w/ P, Q, R as above: True or False: $x = 2$ is an ordinary point? **True**

Ex 3: As above: $x = -2$ is an ordinary pt? **False.**

Thm (Existence of Power Series Solutions about ord. pts.)

If x_0 is an ordinary point of the differential equation,

$$P(x)y'' + Q(x)y' + R(x)y = 0, \quad (*)$$

then:

① There is a power series solution $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ to the ODE. (*).

② The solution from ① can be written as

$$y = a_0 y_1 + a_1 y_2$$

where y_1 and y_2 are two power series solutions to the ODE (*) about x_0.

③ y_1 and y_2 are a F.S.S. (i.e. $W(y_1, y_2) \neq 0$ for some value x).

④ Radius of convergence for both y_1 and y_2 are $\geq$ the minimum of the radii of convergence for $\frac{Q}{P}$ & $\frac{R}{P}$.

Ex: From before, $y'' + y = 0$ has a series solution ($P=1$, $Q=0$, $R=1$)

$$y = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}_{y_1} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{y_2}.$$

By this theorem, we know:

(a) $W(y_1, y_2) \neq 0$ everywhere (actually: $\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0$ everywhere)

(b) y_1, y_2 each solve the ODE

(c) $R.C.(y_1), R.C.(y_2) \geq \min\left(R.C.\left(\frac{Q}{P}\right), R.C.\left(R/P\right)\right)$ where P, Q, R as above

Ex: Find a lower bound for R.C. of series solutions about $x_0 = 0$ for the Legendre Equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

Note: • $P(x) = 1-x^2$, $Q(x) = -2x$, $R(x) = \alpha(\alpha+1)$ all polys.

• $y'' - \underbrace{\frac{2x}{1-x^2}}_{\text{undefined at } x=\pm 1} y' + \underbrace{\frac{\alpha(\alpha+1)}{1-x^2}} y = 0$

• $\frac{2x}{1-x^2}$ has power series about 0 for $-1 < x < 1 \leadsto |x| < 1$

$\frac{\alpha(\alpha+1)}{1-x^2}$ . . . . $\leadsto |x| < 1$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ which converges AT LEAST on $|x| < 1$.

There is a solution

Ex: $(1+x^2)y'' + 2xy' + 4x^2 y = 0$

$\Rightarrow y'' + \underbrace{\frac{2x}{1+x^2}}_{p} y' + \underbrace{\frac{4x^2}{1+x^2}}_{q} y = 0$

(Note: By ch. 3, any IVP w/ this ODE has a solution on $-\infty < x < \infty$ b/c p, q both continuous everywhere!)

Consider $x_0 = 0$: p, q have power series except at $\pm i$, and $d(0, \pm i) = 1$ [$\leftarrow$ Fact I'll give!]. Hence, this ODE has power series soln $y = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < 1$ AT LEAST!