§ 5.2 – Series Solutions near an ord. pt.

Ex: $y'' + y = 0 \longrightarrow$ $\boxed{\text{old way}}$: $r^2+1=0 \Rightarrow r = \pm i$

$\Rightarrow$ gen soln: $y = c_1\cos(x) + c_2\sin(x)$.

New way: Suppose $\sum_{n=0}^{\infty} a_n x^n = y$ is a solution. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \left(= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \right)$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad \left(= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \right),$$

and b/c ODE is $y''+y=0$, we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Longleftrightarrow \quad \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Longleftrightarrow \quad \sum_{n=0}^{\infty} x^n \left[(n+2)(n+1) a_{n+2} + a_n \right] = 0. \qquad (\star)$$

Fact: Power series $=0 \Longleftrightarrow$ every term $=0$, so

$(\star) \Longleftrightarrow (n+2)(n+1)a_{n+2} + a_n = 0$ for all n. $(\star\star)$

Now: we use $(\star\star)$ to figure out what the coefficients must be!
the a_n...

$\longrightarrow$

(Cont'd)

$$(n+2)(n+1)a_{n+2} + a_n = 0 :$$

"a_0"

"a_1"

- $n=0 \Rightarrow 2a_2 + a_0 = 0 \Leftrightarrow a_2 = \frac{-a_0}{2}$
- $n=1 \Rightarrow 3(2)a_3 + a_1 = 0 \Leftrightarrow a_3 = \frac{-a_1}{3(2)}$
- $n=2 \Rightarrow 4(3)a_4 + a_2 = 0 \Leftrightarrow a_4 = \frac{-a_2}{4(3)} = \frac{+a_0}{4(3)(2)}$
- $n=3 \Rightarrow 5(4)a_5 + a_3 = 0 \Leftrightarrow a_5 = \frac{-a_3}{5(4)} = \frac{a_1}{5(4)(3)(2)}$

So: $a_{2k} = \frac{(-1)^k a_0}{(2k)!}$ & $a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}$

$$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

$$= a_0 + a_1 x + \left(\frac{-a_0}{2}\right)x^2 + \left(\frac{-a_1}{3(2)}\right)x^3 + \left(\frac{a_0}{4(3)(2)}\right)x^4 + \left(\frac{a_1}{5(4)(3)(2)}\right)x^5 + \dots$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n)!} x^{2n}}_{\text{power series for cosine}} + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n a_1}{(2n+1)!} x^{2n+1}}_{\text{power series for sine}}$$

So, as expected, $y = c_1\cos(x) + c_2\sin(x)$ [where $c_1 = a_0$ & $c_2 = a_1$] is the solution!

- ~~want to also test them for convergence using ratio test.~~
  ~~"cosine part:" $\lim_{n\to\infty} \left| \frac{(-1)^{n+1} a_0 x^{2n+2}}{(-1)^n a_0 x^{2n}} \right| = \lim_{n\to\infty} |x^2| = |x^2|$~~

2].

want to check for convergence using ratio test:

- "Cosine part": $\lim\limits_{n\to\infty} \left| \dfrac{(-1)^{n+1} a_0 x^{2(n+1)}}{(2n+1)\,b} \cdot \dfrac{(2n)!}{(-1)^n a_0 x^{2n}} \right|$

$$= \lim_{n\to\infty} \left| \frac{1}{2n+1} x^2 \right|$$

$$= |x^2| \cdot \lim_{n\to\infty} \left| \frac{1}{2n+1} \right| = 0 < 1 \text{ always!}$$

$\Rightarrow$ converges abs. for all x.

- "sine part": $\lim\limits_{n\to\infty} \left| \dfrac{(-1)^{n+1} a_0 x^{\overbrace{2(n+1)+1}^{2n+3}}}{\underbrace{(2(n+1)+1)!}_{(2n+3)!}} \cdot \dfrac{(2n+1)!}{(-1)^n a_0 x^{2n+1}} \right|$

$$= \lim_{n\to\infty} \left| \frac{1}{(2n+3)(2n+2)} \cdot x^2 \right|$$

$$= |x^2| \cdot \lim_{n\to\infty} \left| \frac{1}{(2n+3)(2n+2)} \right| = 0 < 1 \text{ always!}$$

So, our solution is abs. convergent <u>everywhere</u>, which means we didn't break any rules by doing this method. (In general, we may not be so lucky!)

Ex: $y'' - xy = 0$.

- Suppose $y = \sum_{n=0}^{\infty} a_n x^n$ sol'n

$$\Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

- So, $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} a_n x^n = 0$

need to match

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{\boxed{n}} - \sum_{n=0}^{\infty} a_n x^{\boxed{n+1}} = 0$$

↕ shift index

match

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{\boxed{n}} - \sum_{n=1}^{\infty} a_{n-1} x^{\boxed{n}}$$

have to match

$$\Rightarrow \underbrace{2a_2}_{\substack{n=0 \\ \text{term}}} + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=1}^{\infty} a_{n-1} x^n \quad \text{for all } n.$$

Coefficients:

- $n=0 \Rightarrow 2a_2 = 0 \Rightarrow a_2 = 0.$ $\left(\Rightarrow (n+2)(n+1) a_{n+2} = a_{n-1}\right)$

not in Σ's →

- $n=1 \Rightarrow 3(2) a_3 = a_0 \Rightarrow a_3 = \frac{a_0}{3!} \Rightarrow a_6 = \frac{a_3}{6(5)} = \frac{a_0}{6\cdot5\cdot3\cdot2}$
- $n=2 \Rightarrow 4(3) a_4 = a_1 \Rightarrow a_4 = \frac{a_1}{4(3)} \Rightarrow a_7 = \frac{a_4}{7\cdot6} = \frac{a_1}{7\cdot6\cdot4\cdot3}$
- $n=3 \Rightarrow 5(4) a_5 = a_2 \Rightarrow a_5 = 0. \Rightarrow a_8 = 0 \ \ldots$

(Cont'd)

So:

$$a_{3n} = \frac{a_0}{2\cdot3\cdot5\cdot6\cdot \dots \cdot(3n-1)(3n)} \qquad a_{3n+1} = \frac{a_1}{3\cdot4\cdot6\cdot7\cdot \dots \cdot(3n)(3n+1)}$$

$$a_{3n+2} = 0$$

$$\Rightarrow y = a_0\left[1 + \frac{x^3}{2\cdot3} + \frac{x^6}{2\cdot3\cdot5\cdot6} + \cdots\right] + a_1\left[x + \frac{x^2}{3\cdot4} + \frac{x^4}{3\cdot4\cdot6\cdot7} + \cdots\right]$$