A (brief) review of series

- For more thorough review, see §5.1 in the text.

Concept Review

- $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ <u>converges</u> at a point x if $\lim_{K\to\infty} \sum_{n=0}^{K} a_n (x-x_0)^n$ exists for that x.
  ↳ Series may converge at some x's and not others. Part of what follows will be figuring out when convergence happens!

- $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ <u>converges absolutely</u> if $\sum_{n=0}^{\infty} |a_n (x-x_0)^n|$ converges.
  ↳ Absolute convergence $\Longrightarrow$ convergence but not conversely (e.g. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n}$ converges by alt. series test but $\sum_{n=0}^{\infty} \left|(-1)^n \left(\frac{1}{n}\right)\right| = \sum_{n=0}^{\infty} \frac{1}{n}$ <u>diverges</u>).

- To test for absolute convergence, you can use the <u>ratio test</u>: $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely at x if

$$\lim_{n\to\infty} \left| \frac{a_{n+1} (x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = |x-x_0| \cdot \underbrace{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}_{L} \left(= |x-x_0| L \right)$$

satisfies $|x-x_0|L < 1$. It diverges if $|x-x_0|L > 1$ & is inconclusive if $|x-x_0|L = 1$.

<u>Ex:</u> $\sum_{n=1}^{\infty} (-1)^n n (x-2)^n \rightsquigarrow \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (n+1) (x-2)^{n+1}}{(-1)^n n (x-2)^n} \right| = \lim_{n\to\infty} \left(\left| \frac{n+1}{n} \right| \cdot |x-2| \right)$

$$= 1 \cdot |x-2| = |x-2|.$$

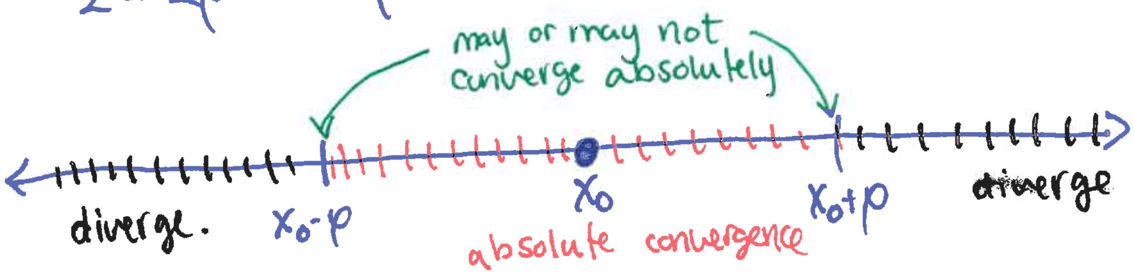
By ratio test, this converges absolutely when $|x-2| < 1 \Rightarrow 1 < x < 3$.
At $x=1$: $\sum_{n=1}^{\infty} (-1)^{2n} n$ diverges; At $x=3$: $\sum_{n=1}^{\infty} (-1)^n n$ diverges. ⟶

- The <u>interval of convergence</u> is the interval on which a series converges absolutely.

  <u>Previous Ex:</u> $(1,3)$

The length of this interval is 2ρ where $\rho =$ <u>radius of convergence</u>.

<u>Previous Ex:</u> Interval $(1,3)$ has length 2. So,
$2 = 2\rho \Rightarrow \rho = 1$ is the radius of convergence.



<u>Ex:</u> Determine the radius/interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot 2^n}$$

- When a series converges absolutely, it represents a <u>function</u>.

For example, on its interval of convergence,

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = f(x) \quad \text{for some } f.$$

We can show that $a_n = \frac{f^{(n)}(x_0)}{n!}$ for all n, and
$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ is called the Taylor (series/) expansion of f.

- Taylor series can be integrated/derived term-by-term:

If $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots,$

then

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^3 + \dots + n a_n x^{n-1} + \dots$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1} \quad = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

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$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

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↑

THIS is going to be used to solve ODEs!